Necessary and Sufficient Conditions for the Sustainability of a Multiproduct Monopoly

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I. Introduction

A recent line of research shows revival of interest in firms and industries that produce more than one good. Special attention is paid to the technological determinants of the structure of multiproduct industries. This focus on multiproduct operations is desirable for two reasons. First, casual empricism suggests that multiproduct firms are the rule rather than exceptions. Second, multiproduct operations involve complex jointness in cost and demand interdependence rendering the standard single product concepts pitifully inadequate for empirical and analytical purposes.

This recent line of research started with the development of new multiproduct cost concepts pertinent to joint production, by Baumol (1976, 1977), Panzar and Willig (1977a, 1978 and 1981), Faulhaber (1975), Sandberg (1975), and Baumol, Bailey and Willig (1977).

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The new multiproduct cost concepts include: economies of scope to describe cost savings from joint production, ray average cost to deal with multiproduct overall returns to scale, product specific economies of scale to capture the impact of variations in a single product holding the remainder of the product set constant, and transray convexity to reflect the tradeoff between cost complementarities and scale effects.

The development of these new multiproduct concepts provided the necessary analytical apparatus for the reexamination of the conventional wisdom underlying the determinants of market structure and, consequently, some surprising results were obtained. Baumol (1977) has shown that economies of scale are neither necessary nor sufficient for a monopoly industry structure and that cost subadditivity, rather the scale economies, is the relevant feature for the analysis of market structure. Faulhaber (1975) has pointed out the fact that a natural monopolist supplying products which are all jointly socially desirable, even if it is producing and pricing efficiently, may find it impossible to set prices which deter entry by uninnovative competitors. The question of the sustainability of a multiproduct industry equilibrium, because of jointness in cost and interdependence in demand, is far more complex than the issue of limit pricing in the single product context. Panzar and Willig (1977b) examine the causes of unsustainability and establish a set of necessary conditions for the sustainability of a multiproduct monopoly. They show that strong demand substitution effect and product specific scale economies work against sustainability while economies of scope favor sustainability. Baumol, Bailey and Willig (1977) provide a sufficient condition. They show that Ramsey-pricing is sufficient for the sustainability of a multiproduct monopoly and that the public interest may be served, even under monopoly, by a variant of the invisible hand.

Examination of the necessary conditions (Panzar and Willig, 1977b) and of the sufficient conditions (Baumol, Bailey and Willig, 1977) reveals that these conditions do not match and that, actually, they are far apart. This article provides a set of conditions which are, simultaneously, necessary and sufficient for the sustainability of a multiproduct monopoly. These conditions include Ramsey-prices as a special case of sustainable prices. The same conditions show that sustainability can be ascertained, as well, for the price-output vectors that are not Ramsey-optimal. Therefore, it is uncertain whether the public interest is served by a monopolist who chooses a sustainable price-output vector in order to deter entry by competitors.

II. Industry Demand and Cost Characteristics

The industry is defined by the set, $N = [1,2,...,n]$, of products supplied by the monopolist. Feasible output vectors are denoted by $Q = (q_1,q_2,...,q_n)$ and the technology is reflected in the multiproduct minimum cost function $C(Q)$. The industry is a potential monopoly characterized by cost subadditivity, i.e., the monopolist's output vector Q^m costs less under a single firm production than under any other alternative.[1] This cost function is continuous and differentiable for $Q > 0$ but it does not rule out discontinuities which are associated with a fixed set-up cost. $C(Q)$ is the monopolist's economic cost function which includes, in the form of rents, all entry costs facing potential entrants. The monopolist's prices vector, $P^m = (p_1^m,p_2^m,...,p_n^m)$, is related to its output vector, Q^m, by means of the system of differentiable market demand functions: $q_i^m = f^i(P^m)$, $i = 1,2,..., n$. Additionally, the following five explicit assumptions will be utilized.

A1

Potential entrants consider supplying only goods in the natural monopoly product set N. Potential entrants and the monopolist have access to the same production technologies and, therefore, they face identical economic cost functions.

A2

(a) the monopolist's cost function exhibits non-increasing ray average cost; and(b) the n-dimensional cost function is weakly convex in every subspace of $n-1$ products or less. That is, the principal minors of order $n-1$ or less, of the matrix of the cross-partials (C_{ij}) of the cost function, are nonnegative.

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1) Throughout this paper we denote variables pertaining to the monopolist and to the new entrant by the superscripts m and e, respectively.

Part (a) of A2 requires that the monopolist enjoys overall economies of scale for proportional variation in all products. Part (b) of A2 does not require the convexity of the cost function in the n-dimensional product space. Actually, (a) rules out a convex total cost function unless sufficient fixed cost is present. Assumption 2(b) leaves the sign of the principal minor of order n indeterminate. This assumption does not preclude cost complementarity or substitutability, nor does it require transray convexity of the cost function.[2] However, it does imply nondecreasing marginal cost for each output. Furthermore, assuming zero product-specific fixed set-up cost, A2 implies product-specific (weak) diseconomies of scale as well as economies of scope; it also implies cost subadditivity.[3]

A3

The demands for the goods in the monopoly product set are (a) Normal; (b) Weak Gross Complements (WGC):

(a) Normality: if either $P^B \neq P^A$ or $Q^B \neq Q^A$ then

$(p_i^B - p_i^A)\ (q_i^B - q_i^A) < 0$ for at least one i, $i \in N$.

This definition of normality states that the relationship between price and quantity is strictly negative for variations in one item (either quantity or price) at the time. Furthermore, it rules out the possibility that all quantities (prices) respond positively to changes in their own prices (quantities) in the event of a simultaneous variation of more than one price (quantity). Finally, it is easy to verify, by contradiction, that the above definition of normality establishes one-to-one correspondence between vectors of quantities of outputs and vectors of market prices.[4]

(b) Weak Gross Complements: if $P^B \geq P^A$ and $P_i^B = P_i^A$ for $i \in V \subset N$, then $q_i(P^B) \leq q_i(P^A)$ for $i \in V$.

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2) Examine, for example, the cost function $C(q_1,q_2) = F + q_1^2 \cdot q_2^2$ which meets our requirement A2(b) since $C_{11} \geq 0$ and $C_{22} \geq 0$. Yet, this cost function has a concave segment along the transray $q_1+q_2=K$ where it obtains a local maximum at $q_1=q_2=K/2$.

3) For the interrelationships among the various multiproduct cost concepts see Baumol (1977) and Willig (1979). Also note that for analytical convenience we assume zero product specific set-up cost though this is not necessary as shown in footnote 12)

4) For proof see lemma 4 in Baumol, Bailey and Willig.

That is, the goods in N are WGC if a rise in the prices of the goods in N - V will never increase the demand for the goods in V. In the special case of variation in one price at the time, WGC implies $\partial q_j / \partial p_i \geq 0$ for $\forall$ i,j where $i \neq j$.

A4

The monopolist's profit function is strictly concave and it has an interior maximum.

Given the monopolist's price and quantity vectors P^m and Q^m, we have, from normality, that whenever $P \neq P^m$ or $Q \neq Q^m$ then (i) $(P_i - P^m_i)(q_i - q^m_i) < 0$ for at least one i. Next, denoting the marginal profit with respect to q_i by π_i, we have, from the strict concavity of the profit function, that whenever $Q \neq Q^m$ then (ii) $(\pi_i - \pi^m_i)(q_i - q^m_i) < 0$ for at least one i. The relationship of inequality (ii) to concavity needs some clarifications. Strict concavity assures that the matrix (π_{ij}) of the cross-partial of the profit function is negative definite and, therefore, Hicksian. A proposition by Gale and Nikaido states that, if the matrix (π_{ij}) is Hicksian, the inequalities $(q_i - q^m_i)(\pi_i - \pi^m_i) \geq 0$ for 1,2,...,n have only the trivial solution[5] $Q = Q^m$. Then our inequality (ii) follows immediately. Our assumption 5 strengthens the inequalities (i) and (ii) as follows:

A5

(a) $(p_i - p^m_i)(q_i - q^m_i) < 0$ for at least one i, where $i \in K$ and $K = \{ i \mid p_i > p^m_i \}$; and

(b) $(p_i - p^m_i)(q_i - q^m_i) < 0$ and $(\pi_i - \pi^m_i)(q_i - q^m_i) < 0$ for at least one i, where $i \in G$ and $G = \{ i \mid q_i > q^m_i \}$.

A5 requires a somewhat stronger normality (negative relationship) in demand and in marginal profit because it stipulates that the inequality (i) must hold for at least one item from the set of increased prices; and that the inequalities (i) and (ii) hold simultaneously for at least one output from the set of increased quantities.

We note that for the special case of a variation in only one output, assumption 5 is not necessary because inequalities (i) and (ii) always hold simultaneously for the increased output as implications of normality and concavity. That is, let $q_i > q^m_i$ be the only element

5) See Takayama p. 281.

of $G = \{i \mid q_i > q_i^m\}$ while $q_j = q_j^m$ for $j \neq i$: then normality implies (i) $(p_i - p_i^m)(q_i - q_i^m) < 0$ for $i \in G$; and utilizing $\pi_{ii} < 0$ from concavity, we obtain (ii) $(\pi_i - \pi_i^m)(q_i - q_i^m) < 0$ for $i \in G$. Thus the output q fulfills simultaneously both inequalities (i) and (ii) Next consider the case of a simultaneous increase in many outputs, i.e., $q_i > q^m$ for $i = 1,2,\ldots,g$ while $q_j = q_j^m$ for $j \neq i$. Then normality implies (i) $(p_i - p_i^m)(q_i - q_i^m) < 0$ for at least one i, where $i \in G$. Now assuming independence in the marginal profit productivity of the various outputs, i.e., $\pi_{lh} = 0$ for $l \neq h$, then concavity implies (ii) $(\pi_i - \pi_i^m)(q_i - q_i^m) < 0$ for all i's and the inequalities (i) and (ii) prevail simultaneously for at least one i which belongs to G. Thus the results of assumption 5 follow naturally from normality and concavity, for a variation in a single product in general, and for an increase in many products when $\pi_{ij} = 0$ for $i \neq j$. Actually assumption 5 restricts the magnitudes of the price and quantity cross-effects relatively to their own-effects so that the own effect dominates for at least one increased item.

Now we turn to the question of the necessary and sufficient conditions for the sustainability of a multiproduct monopoly under the above assumptions A1 through A5. For this purpose we adopt the definition of sustainability used in Panzar and Willig (1977b) and Baumol, Bailey and Willig (1977).

Definition: A stationary monopoly price-output vector combination, $P^m - Q^m$, is sustainable if new entrants anticipate only negative profits from supplying any subset of the industry product set N at prices $p_i^e \leq p_i^m \; \forall \; i$ and $p_i^e < p_i^m$ for at least one i. It is also required that the monopolist's profit, valued at $P^m - Q^m$, is nonnegative.[6)]

Proposition

The following conditions are necessary and sufficient for the monopoly price-output vector combination, $P^m - Q^m$, to be sustainable:[7)]

6) Actually, Panzar and Willig (1977b) require $p_i^e < p_i^m$ for all i in order to avoid the ambiguity as to how demand splits between the incumbent and the new entrant when $p_i^e = p_i^m$ However, if the new entrant enjoys positive profits with $p_i^e < p_i^m$ for at least one i then, by continuity, he can generate positive profits with $p_i^e < p_i^m$ for all i.

7) Panzar and Willig (1977b) show a set of necessary conditions which include the above conditions (1) and (3).

(1) $\pi^m = 0$

(2) $MR_i^m \leq MC_i^m$, $i = 1,2,\ldots,n$

(3) $P_i^m \geq MC_i^m$, $i = 1,2,\ldots,n$

III. Necessary Conditions for Sustainability

Proof of necessary condition (1): $\pi^m < 0$ is infeasible because it calls for the exit of the monopolist from the industry. If $\pi^m > 0$, then the new entrant can supply Q^m at $P^e \equiv (p_1^e, \cdots\cdots p_n^e) = (p_1^m - \varepsilon, \cdots, p_n^m - \varepsilon)$ with $\pi^e = \sum_i^n p_i^e q_i^m - C(Q^m) = \pi^m - \varepsilon \sum_i^n q_i^m \geq 0$ by choosing ε such that $0 < \varepsilon \leq \pi^m / \sum_i^n q_i^m$. Therefore, both $\pi^m < 0$ and $\pi^m > 0$ are ruled out and $\pi^m = 0$ is necessary.

Necessary condition (2): Marginal revenues must not be above marginal costs. Proof: We first prove lemma 1 and then utilize it to establish the necessary condition (2).

Lemma 1: If $q_i^B > q_i^A \ \forall\, i$ then $p_i^B < p_i^A \ \forall\, i$.

Proof: Suppose

$$q_i^B > q_i^A \ \forall\, i \text{ but } p_i^B > p_i^A \text{ for some i's.}$$

Then,

$$p_i^B > p_i^A \text{ for some i's.}$$

assures, via assumption 5, that

$$q_i^B < q_i^A \text{ for at least one i.}$$

This contradicts the initial assumption that $q_i^B > q_i^A \ \forall\, i$. Thus, if $Q^B > Q^A$ then $P^B < P^A$.

Next, suppose that, in contradiction to necessary condition (2),

$$MR_j^m > MC_j^m \text{ for } j \in J$$

and

$$MR_k^m \leq MC_k^m \text{ for } k \in N - J.$$

Then consider new entry at $Q^e = Q^m + dQ$, where $dQ > 0$, so that $Q^e > Q^m$ implies $P^e < P^m$

due to lemma 1. Since the monopolist's prices are not binding for $dQ > 0$, the new entrant faces the original revenue function of the industry where $d\pi^e = \sum_i^n (MR_i - MC_i)\, dq_i$ for $dQ > 0$. Therefore the new entrant can choose $dQ = (dq_1, ..., dq_n) > 0$ such that

$$\sum_j (MR_j^m - MC_j^m)\, dq_j > -\sum_k (MR_k^m - MC_k^m) dq_k$$

or, equivalently,

(4) $$d\pi^e = \sum_j \pi_j\, dq_j + \sum_k \pi_k\, dq_k = \sum_{i=1}^n \pi_i^m dq_i > 0,$$ where π_i is the partial derivative of the profit function with respect to q_i. Hence, starting at $\pi^m = 0$ and obtaining $d\pi^e > 0$ means that profitable entry is feasible unless $MR_i^m \leq MC_i^m \; \forall i$.

The geometrical interpretation of the necessary conditions $MR_i^m \leq MC_i^m$ is straightforward in the case of two products. Consider the projection of the positive portion of the monopolist's profit function into the $q_1 - q_2$ plane. Figue (1) shows this projection of $\pi \geq 0$ as a convex set because, the profit function is assumed to be strictly concave and, the $\pi = 0$ locus (the boundary of the set) is an isoprofit of the strictly concave profit function. Now, suppose the monopolist chooses the output vector Q^m on the positively sloped segment of the zero profit locus. Analytically, since $d\pi = 0$ defines the isoprofit curve, the slope of the zero profit locus at Q^m is obtained by solving $d\pi = 0$ for $dq_2/dq_1 = -(MR_1^m - MC_1^m)/(MR_2^m - MC_2^m)) > 0$ so that either $MR_1^m > MC_1^m$ or $MR_2^m > MC_2^m$. Thus our conditions $MR_i \leq MC_i$ are violated and the new entrant has access to the profitable region by choosing $Q^e > Q^m$ which, as we have shown, implies $P^e < P^m$. Since the monopolist's prices are not binding at Q^e, it follows that the new entrant faces the original profit function of the monopolist with positive profits at Q^e. Our conditions $MR_i \leq MC_i$ restrict sustainable output vectors Q^m to the negative sloped segment of the zero profit locus. Actually, conditions (1), (2) and (3) restrict sustainable output vectors Q^m to the nonpositively sloped segment in the north-eastern portion of the zero profit locus.

Necessary condition (3): Prices must not be below marginal costs. The following proof is provided by Panzar and Willig (1977b).

Proof: An entrant attempts to market $Q^e = Q^m - \Delta q_i$ at P^m. Sustainability requires $P^m \cdot Q^e < C(Q^e)$, i.e., $P^m(Q_m - \Delta q_i) < C(Q^m - \Delta q_i)$. But $C(Q^m) \leq P^m Q^m$, so it is necessary that $P_i^m \Delta q_i > C(Q^m) - C(Q^m - \Delta q_i)$ and $p_i^m > [C(Q^m) - C(Q^m - \Delta q_i)]/\Delta q_i$. Taking the limit as $\Delta q_i \to 0$, yields $p_i^m \geq \partial C(Q^m)/\partial q_i$

IV. Sufficient Conditions for Sustainability

Now we turn to show that conditions (1), (2) and (3) are sufficient for sustainability. For this purpose we, first, define the relevant sets and notations. Let S denote the entire output space, i.e.,

$$S = \{ Q | q_i \geq 0 \text{ for all } i,\ i = 1,2,\dots,n \},$$

where Q is the n-component output vector $Q = (q_1, q_2, \dots, q_n)$. Then we partition S into two mutually exclusive and exhaustive subsets,

$$S^- = \{ Q | q_i \leq q_i^m \text{ for all } i,\ i = 1,2,\dots,n \}$$

and

$$S^+ = \{ Q | q_i > q_i^m \text{ for at least one } i \},$$

where $S^- \cup S^+ = S$ and $S^- \cap S^+ = \phi$. Finally, we define the boundary, B, of the set S as

$$B = \{ Q^b | q_i^b \leq q_i^m \ \forall i \text{ and } q_i^b = q_i^m \text{ for at least one } i,\ i = 1,2,\dots,n \},$$

where $Q^b = (q_1^b, q_2^b, \dots, q^b)$ represents any arbitrary output vector on the boundary of S^-.

Our sufficiency proof involves three distinct steps. First we prove that new entry on the boundary of the set S^- is not profitable, i.e., $\pi^e(Q^b) < 0$ for $Q^b \in B$. Then, using the boundary as a reference, it is shown in step 2 that entry in S^+ is not profitable. That is, we show that $\pi^e(Q) < \pi^e(Q^b) < 0$ for $Q \in S^+$. Finally, the third step shows that for $Q \in S^-$ we have $\pi^e(Q) < \pi^e(Q^b) < 0$, i.e., entry within the set S^- is infeasible.

Step 1: Entry in B is not profitable.

Proof: Consider a boundary output vector Q^b such that

$$q_i^b < q_i^m \text{ for } i = 1,2,\dots,x$$

and

$$q_i^b = q_i^m \text{ for } i = x+1,\ x+2,\dots,n,$$

where $x < n$ and $q_i^b = q_i^m$ for at least one i. Then the pseudoprofit at Q^b is given by

(5) $\pi^b(Q^b) = H(Q^b) - C(Q^b) = \sum_{i=1}^{x} p_i^m q_i^b + \sum_{i=x+1}^{n} p_i^m q_i^m - C(Q^b)$

where $H \equiv \sum_i^n p_i^m q_i$ is the pseudorevenue function.[8] Next we apply the Taylor expansion to obtain the following expansion of $\pi^P(Q)$ around the point Q^m:

(6) $$\pi^b(Q^b) = \pi^p(Q^m) + \frac{\partial \pi^p}{\partial Q}(Q^m)[Q^b - Q^m] + \frac{1}{2}[Q^b - Q^m]\pi^p_{ij}(\eta)[Q^b - Q^m]$$

where $\pi^P(Q^m) = H(Q^m) - C(Q^m) = 0$ from the necessary condition $\pi(Q^m) = 0$. Moreover, the second term of equation (6) is

(7) $$\frac{\partial \pi^p}{\partial Q}(Q^m)[Q^b - Q^m] = \sum_{i=1}^{x}(p_i^m - MC_i^m)(q_i^b - q_i^m) \leq 0$$

since $(q_i^b - q_i^m) < 0$ by definition and $(p_i^m - MC_i^m) \geq 0$ from the necessary conditions (2). Finally, the last term of (6),

(8)

$$[Q^b - Q^m]\pi^p_{ij}[Q^b - Q^m]$$

$$= [(q_1^b - q_1^m), \cdots\cdots, (q_x^b - q_x^m)](-1)\begin{bmatrix} C_{11}, & C_{12}, & \cdots\cdots\cdots, & C_{1x} \\ C_{21}, & C_{22}, & \cdots\cdots\cdots, & C_{2x} \\ \vdots & \vdots & & \vdots \\ C_{x1}, & C_{x2}, & \cdots\cdots\cdots, & C_{xx} \end{bmatrix}\begin{bmatrix} q_1^b - q_1^m \\ \vdots \\ q_x^b - q_x^m \end{bmatrix} \leq 0$$

is a negative semidefinite quadratic form because the above matrix $[C_{ij}]$ is an x-dimensional $(x < n)$ principal minor of the n-dimensional matrix of the cross-partials of the cost function, (C_{ij}), and as such it is positive semidefinite by assumption[9] 2(b). Thus the first term of equation (6) is zero while the last two terms are nonpositive and, therefore, the pseudoprofit function takes only nonpositive values along the boundary, i.e., $\pi^P(Q^b) \leq 0$ for $Q^b \in$ **B**. However, the new entrant has to choose a vector of prices $P^e \leq P^m$ where $p_i^e < p_i^m$ for at least one i. Then the revenue of the new entrant must be below the pseudorevenue, $R^e < H$,

8) Baumol, Bailey, and Willig have labeled H 'pseudorevenue' because the prices P_i^m are held fixed while quantities vary freely without regard to demand conditions. Thus the pseudorevenue differs from the market revenue, though they must coincide at Q^m

9) There is no need to assume that A2(b) holds globally. Our proof requires A2(b) to hold only along the boundary, i.e., for $Q \in B$.

and

$$\pi^{e}(Q^{b}) < \pi^{p}(Q^{b}) \leq 0,$$

i.e., profitable entry on the boundary of the set S^{-} is infeasible.

Step 2: Entry with $Q \in S^{+}$ is not profitable. First we prove lemma 2 and then we utilize this lemma to show that profitable entry in S^{+} is infeasible.

Lemma 2: A5(a) and A3(b) together imply that $\partial p_j / \partial q_i \geq 0$ for all i and j, j ≠ i.

Proof of lemma 2: Here we show that when the demand of only one good, q_i, increases then no price of any other good decreases, i.e., all crossprice effects with respect to output are nonnegative. Suppose

$$q_i^B > q_i^A, \quad q_j^B = q_j^A \text{ for } j \neq i.$$

and, in contradiction to lemma 2,

$$p_j^B < q_j^A \text{ for at leat one } j, \quad j \neq i.$$

Now define a new price vector P^C,

$$P^C = \{ p_i^C = p_i^A \ ; \ p_j^C = p_j^B \text{ for } j \neq i \}.$$

Then compare the price vectors P^C and P^B to obtain

$$p_j^C = p_j^B \text{ for } j \neq i.$$

and

$$p_i^C > p_i^B \text{ because normality and } q_i^B > q_i^A \text{ imply } p_i^B < p_i^A.$$

Thus $P^C \geq P^B$, and the WGC assumption A3(b) implies

$$q_j^C \leq q_j^B \text{ for } j \neq i.$$

Next compare the price vectors P^A and P^C to verify that

$$p_i^A = p_i^C$$

and, by supposition,

$$p_j^A > p_j^C \text{ for at least one j, j} \neq \text{i, because } p_j^C = p_j^B.$$

Therefore, A5(a) implies

$$q_j^A < q_j^C \text{ for at least one } j, \ j \neq i.$$

The above two comparisons reveal that $q_j^A < q_j^B$ for at least one j, $j \neq i$. But this contradicts the initial supposition and so it is proven that under A3(b) and A5(a) $\partial p_j/\partial q_i < 0$ for at least one j, $j \neq i$, is impossible. Therefore, $\partial p_j/\partial q_i \geq 0$ for all i, j where j = i.

Proof that entry in S^+ is not profitable: Suppose the new entrant attempts to sell $Q^e \in S^+$. Then the set $G = \{ i | q_i^e > q_i^m \}$ is not empty by the definition of S^+. Now we define the set L as $L = \{ i | p_i < p_i^m;\ i = 1,2,\ldots, l \}$. Then A5(b) assures that the intersection of these two sets is not empty, i.e., $G \cap L \neq 0$. Thus the new entrant's effective profit function is

$$(9) \qquad \pi^e(Q^e) = \sum_{i=1}^{l} p_i q_i^e + \sum_{i=1-l}^{n} p_i^m q_i^e - C(Q^e)$$

Differentiation with respect to q_i^e, for $i \in G \cap L$, yields

$$(10) \qquad \partial\pi^e/\partial q_i^e = p_i + \sum_{j=1}^{l} q_j^e\, \partial p_i/\partial q_i^e - MC_i \leq \partial\pi/\partial q_i^e$$

$$= p_i + \sum_{j=1}^{n} q_j^e\, \partial p_j/\partial p_i^e - MC_i,$$

where the above inequality follows from the fact that $\partial p_j/\partial q_i \geq 0$ by lemma 2. That is, lemma 2 ascertains that the new entrant's effective marginal revenue $[MR_i^e(Q^e)]$ is below the original marginal revenue $[MR_i(Q^e)]$. However, under the necessary conditions $\pi_i(Q^m) = MR_i^m - MC_i^m \leq 0$, A5(b) assures that $\pi_i(Q^e) < \pi_i(Q^m) \leq 0$ for at least one $i \in G \cap L$. Therefore, equation (10) can be restated as

(11)

$$\pi_i^e(Q^e) = MR_i^e - MC_i \leq \pi_i(Q^e) = MR_i - MC_i < 0 \quad \text{for at least one } i,\ i \in G \cap L.$$

Hence, the new entrant can always increase π^e by decreasing q_i^e for $i \in G \cap L$, i.e., by moving towards the boundary of the set S^-. This result, when combined with our proof that entry on the boundary is not profitable, ascertains that for every $Q^e \in S^+$ there exists a $Q^b \in B$ such that $\pi^e(Q^e) < \pi^e(Q^b) < 0$. That is, S^+ is dominated by the negative-profit output vectors along B.

Step 3: Entry with $Q \in S^-$ is not profitable. We have already shown that

$$H(Q^b) = \sum_{i=1}^{n} p_i^m q_i^b < C(Q^b) \;\forall\; Q^b,\ Q^b \in B$$

Then

(12) $H(Q^b)/Q^{-b} = \sum_{i=1}^{n} \alpha_i^b p_i^m \leq C(Q^b)/Q^{-b} \ \forall\, Q^b,$

where $Q^{-b} = \sum_i^n q_i^b$ and $\alpha_i^b = q_i^b / Q^{-b}$. That is, ray pseudoprice is no greater than ray average cost (RAC). Now consider entry at Q^e along the ray passing through Q^b, i.e., $Q^e = \mu Q^b$ with $0 \leq \mu \leq 1$. The assumption of nonincreasing ray average cost, A2(a), yields

$$RAC(Q^e) \geq RAC(Q^b) \ \forall\, Q^b,\ Q^e = \mu Q^b,\ 0 \leq \mu \leq 1,$$

while

$$H(Q^e)/Q^{-e} = H(Q^b)/Q^{-b} \ \forall\, Q^e = \mu Q^b$$

where $Q^{-e} = \sum_i^n q_i^e = \mu Q^{-b}$. These two relations together with the inequality (12) restrict the sign of the pseudoprofit function,[10)]

(13) $\pi^p(Q^e) = [H(Q^e)/Q^{-e} - RAC(Q^e)]\ Q^{-e} \leq 0, \ \forall\, Q^e = \mu Q^b$

However, the new entrant must charge $p_i^e \leq p_i^m \ \forall$ with $p_i^e < p_i^m$ for at least one i. Therefore, $R^e(Q^e) < H(Q^e)$ and

$$\pi^e(Q^e) < \pi^p(Q^e) \leq 0 \ \forall\, Q^e,\ Q^e \in S$$

Thus steps 1 and 3 show that profitable entry in S^- is infeasible while step 2 rules out profitable entry in S^+. Hence, entry in $S = S^- \cup S^+$ yields negative profits and the price-output combination $P^m - Q^m$, meeting the conditions $\pi^m = 0$, $MR_i^m \leq MC_i^m$ and $p_i^m \geq MC_i^m$, is sustainable.

A geometrical interpretation of the sufficiency conditions, specialized to the case of two products, is provided in figure (2). Figure 2(a) shows the projection of the $\pi \geq 0$ region into the $q_1 - q_2$ plane. The projection of $\pi \geq 0$ is a convex set because $\pi = 0$ is an isoprofit locus of a strictly concave profit function. The monopolist's output vector $Q^m = (q_1^m, q_2^m)$ satisfies the conditions $\pi^m = 0$, $p_i^m \geq MC_i^m \geq MR_i^m$ for i = 1,2. The set S^- contains all

10) First it is easy to verify, at this stage, that cost subadditivity follows from A2. Consider the output vectors $Q^k \geq 0$ where $\sum_k Q^k = Q^m$ so that $Q^k \in S^-$. Utilizing A2, we have obtained equation (13) which ascertains that $H(Q^k) \leq RAC(Q^k) \cdot \bar{Q}^k = C(Q^k)$. Summation yields $H(Q^m) = \sum_k H(Q^k) \leq \sum_k C(Q^k)$. But $H(Q^m) = C(Q^m)$ and, therefore, $C(Q^m) \leq \sum_k C(Q^k)$. Second, our proof that $\pi(Q^e) < 0$ for $Q^e \in S^+$ shows that A5 need hold only for the intersection between S^+ and the set of positive-profit output vectors.

output combinations included in the rectangular $q_1^m 0 q_2^m Q^m$. S^+ is the set of all nonnegative output pairs outside the rectangular $q^m 0 q^m Q^m$. The boundary of the set S^*, B, is the set of output vectors lying along the two line segments along $q_1^m Q^m q_2^m$. Now a competitor attempts entry along the boundary at Q^b. Then the behavior of the pseudorevenue plane (H) and the total cost surface (C) along the line segment $q_1^m Q^m$ (boundary) is shown in figure 2(b). The necessary conditions $\pi^m = 0$ and $p_2^m \geq MC_2^m$ ascertains that, at Q^m,$H = p_1^m q_1 + p_2^m q_2 = C$ and $\partial\pi/\partial q_2 = p_2^m \geq MC_2 = \partial C/\partial q_2$. That is, the pseudorevenue intersects the cost surface from below at Q^m. But A2(b) assures that the cost surface is convex along the boundary,[11] i.e., $C_{22} \geq 0$. Therefore, $\partial^2 C/\partial q_2^2 \geq 0 = \partial^2 H/\partial q_2^2$ implies that C lies above H over the entire line segment $q_1^m Q^m$ (unless $\partial^2 C/\partial q_2^2 = 0$ which means that H and C coincide). Similar argument applies to the line segment $q_2^m Q^m$ and thus, along the boundary, C is never below[12] H. But the new entrant cannot charge higher prices (and at least one price must be strictly less) than the monopolist's prices so that $R^e = p_1^e q_1^e + p_2^e q_2^e < H = p_1^m q_1^e + p_2^m q_2^e \leq C(Q^e)$ and profitable entry along the boundary is infeasible.

Next consider entry at $Q^+ \in S^+$ in figure 2(a). Q^+ lies in the potentially profitable region where the pseudorevenue plane is above the cost surface (H = C is the intersection between the pseudorevenue and cost). A vertical movement from Q^b through Q^+ in figure 2(a) corresponds to movements along the horizontal axis in figure 2(c). In this latter figure the original revenue function (R) and the pseudorevenue function are above total cost at Q^+ and profitable entry is not ruled out. However, the new entrant faces an effective revenue function, R^e, which is below the original market revenue function as well as below the pseudorevenue. Actually, A3(a) implies $\partial p_2/\partial q_2 < 0$ while lemma 2 guarantees $\partial p_1/\partial q_2 \geq 0$ so that at Q^b (where $q_1^b = q_1^m$ and $q_2^b < q_2^m$) the pair of prices derived from the original demand function must satisfy $p_1 \leq p_1^m$ and $p_2 > p_2^m$. By the same argument $\partial p_1/\partial q_1 < 0$

11) Transray convexity appears to be more restrictive than our cost assumption. We require weak cost convexity only for movements parallel to the axis while transray convexity requires a convex cost structure along every negatively sloped hyperplane.

12) Though figure 2(b) assumes the absence of product-specific fixed set-up cost, F_2, it is obvious that the presence of F_2 leaves our conclusions intact provided that $F_2 \leq (C-H)$, where C and H are evaluated in the neighborhood $(q_1^m, 0)$.

and $\partial p_2/\partial q_1 \geq 0$ and, therefore, $p_1 < p_1^m$ and $p_2 > p_2^m$ at Q^+ (where $q_1^+ > q_1^b$ and $q_2^+ = q^b$). Thus the new entrant faces the prices $p_1^e = p_1$ and $p_2^e = p_2^m$ for vertical movements from Q^b through Q^+. Equivalently, the new entrant faces the effective revenue function $R^e = p_1 q_1 + p_2^m q_2^b$ for movements parallel to the q_1-axis as shown in figure 2(c). The origin of this figure is a boundary point where $R(Q^b) > C(Q^b) \geq H(Q^b) > R^e(Q^b)$ and $\pi^e(Q^+) < 0$. Moreover, every point Q^+ to the right of Q^b has the property $\pi_1(Q^+) < 0$ (where π_1 is the partial derivative of the original profit function with respect to q_1) which follows from A5(b). This assures that $MR_1(Q^+) < MC_1(Q^+)$, i.e., the original revenue curve (R) is always flatter than the cost curve (C) for $q_1^+ > q_1^b$. But A5(a) and A3(b) assure that $\partial p_2/\partial q_1 \geq 0$ with the consequence of $MR_1^e(Q^+) \leq MR_1(Q^+)$. Hence the new entrant's effective revenue curve (R^e) is always flatter than the original revenue curve and the cost curve, i.e., $MR_1^e(Q^+) \leq MR_1(Q^+) < MC_1(Q^+)$.

Since $R^e(Q^b) < C(Q^b)$ and $MR_1^e(Q^+) < MC_1(Q^+)$ for all vectors $Q^+(q_1^+ > q_1^b, q_2^+ = q_2^b)$ it follows that the excess cost over effective revenue is increasing with respect to q_1 and $\pi^e(Q^+) = R^e - C < 0$ for all Q^+ to the right of Q^b.

Finally, consider attempted entry within the rectangle $q_1^m O q_2^m Q^m$ with an output vector $Q^- = \mu Q^b$, where $0 \leq \mu < 1$, as shown in figure 2(a). We have already shown that $H(Q^b) \leq C(Q^b)$ and $\pi^e(Q^b) < 0, \forall Q^b$. Moreover, the ray pseudoprice $(\sum_i^n \alpha_i^b p_i^m)$ is independent of the scale factor μ while the ray average cost (RAC) is, by A2(b), nonincreasing with respect to μ. Combining the facts that at Q^b we have $\sum_i^n \alpha_i^b q_i^m \leq$ RAC with $\partial RAC(\mu Q^b)/\partial\mu \leq 0$, yields the result that ray average cost is above ray pseudoprice over the entire line segment OQ^b along the ray μQ^b in figure 2(a). The same argument and the same result apply to the line segment OQ^m or, as a matter of fact, to every line segment connecting the origin to an output vector on the boundary of S^-. This means, as shown in figure 2(d), that along the entire line segment OQ^b (or OQ^m) cost is never below pseudorevenue.[13] Therefore $H(Q^-) \leq C(Q^-)$ for all $Q^- \in S^-$ and $R^e(Q^-) < H(Q^-)$ guarantee $\pi^e(Q^-) < 0$ for all $Q^- \in S^-$.

13) Figure 2(d) shows that the cost surface along each ray is concave. This is not necessary when fixed costs are present. It is required that RAC be non-increasing, and this requirement can be satisfied under a large variety of shapes of the total cost along a ray.

V. Sustainability and Ramsey Optimality

The nature of the relationship between sustainability conditions and social welfare is an important issue in the sustainability literature. Baumol, Bailey, and Willig (1977) have shown that the monopolist, in order to be sustainable, is likely to choose Ramsey prices, i.e., prices (and outputs) which are Pareto optimal subject to the constraint that the firm earns profit equal to the maximal economic profit allowed by barriers to entry. Specifically, first they show that, under their set of assumptions, price-output vectors satisfying the Ramsey-pricing rule are sufficient to guarantee sustainability. Second, they argue that it is generally uncertain whether a price-output vector that is not Ramsey-optimal would be sustainable; global information about the demand and cost functions is needed in order to resolve this uncertainty concerning the sustainability of the monopoly. In sum, the above authors conclude that the power of the invisible hand, traditionally limited to perfect competition, has been underestimated. The invisible hand may be potent over a monopoly because Ramsey-pricing provides safety from competitive entry and, conversely, the monopolist may be vulnerable to potential entry if its decisions are not in line with the Ramsey rule.

Baumol, Bailey, and Willig have produced a plausible invisible hand argument for monopolistic markets, based on the interrelationship between sustainability and Ramsey prices. The authors refer to their result as a *weak* invisible hand for two good reasons.

First, the requirements for sustainable Ramsey prices are quite restrictive. The monopolist must find an output vector Q^m such that the pseudorevenue hyperplane will support the zero-profit locus at Q^m in the north-eastern portion of this locus. Note that, generally, the supporting hyperplane at a particular point will differ from the pseudorevenue hyperplane corresponding to that point. Graphically, we can fit a supporting hyperplane, $T = \sum_i^n t_i q_i$, through Q^m such that the entire positive profit region lies above it and the sole point of contact is the tangency between the line T = C and the $\pi = 0$ locus, as shown in figure 2(a). This supporting hyperplane differs from, and it actually interescts, the pseudorevenue hyperplane $(H = \sum_i^n p_i^m q_i)$ at Q^m. The monopolist can announce the prices t_i and prevent entry. However, since the pseudorevenue (H) and the supporting hyperplane (T) differ, the announced prices, t_i, deviate from the market-clearing prices, p_i, and the monopolist fails to sell a portion of the output vector Q^m, which results in negative profits. If the monopolist is to realize zero profit and prevent entry, his choice must be a vector Q^m such that the supporting hyperplane (T) coincides with the pseudorevenue (H) so that the t_i 's happen to satisfy the inverse demand functions $t_i = p_i(Q^m)$. The existence of such a supporting hyperplane, where the t_i's clear the market at Q^m, is not assured in general. Yet, T and H must coincide to produce sustainable Ramsey prices. Moreover, even when T coincides with H to generate a supporting pseudorevenue at Q^m, the resulting prices are not necessarily Ramsey prices. It is further required that the entry function be constant (independent of Q) along the north-eastern segment of the $\pi = 0$ locus, and the authors assume this to be the case. Finally, assuming T coincides with H and constant entry cost, still the resulting prices at Q^m might deviate from the Ramsey rule. This happens when Q^m is the maximum-profit as well as the zero-profit output,[14] where $\pi(Q^m) = 0$, $MR_i^m = MC_i^m$ and $p_i^m \geq MC_i^m \; \forall i$. It is obvious that the maximum-profit zero-profit price-output vector is sustainable in spite of the fact that it rules out Ramsey-pricing.

Second, the above authors do not claim that Ramsey prices are necessary for sustainability. Their point is that it is uncertain whether prices which are not in accord with the Ramsey rule would be sustainable, and that this uncertainty might induce the monopolist to choose Ramsey prices. However, our conditions of $\pi^m = 0$, $MR_i^m \leq MC_i^m$ and $p_i^m \geq MC_i^m$, were shown to be necessary and sufficient for sustainability. These conditions allow the monopolist to choose prices such that the pseudorevenue actually intersects the profitable region rather than supports it at Q^m. Moreover, the entry cost function need not be constant with respect to output, and the maximum-profit zero-profit point does not have to be ruled out for sustainability purposes. The above conditions include the price-output vector satisfying the Ramsey-pricing rule as a special case of sustainable prices. Yet, our conditions

14) There is no good economic reason for the a priori exclusion of the maximum-profit zero-profit point. It is possible that the general equilibrium determination of the entry cost function will lead to that point for sustainability purposes.

show that sustainability can be ascertained as well, for price-output vectors that are not Ramsey-optimal. Therefore, there is no incentive for the adoption of Ramsey prices and the weak invisible hand is further weakened.

VI. Conclusion

We have derived a set of conditions which are simultaneously necessary and sufficient for the sustainability of a monopoly. Since Ramsey-prices are only a special case of these conditions, it cannot be inferred that the public interest is served, in general, by a monopolist who chooses a sustainable price-output vector.

Though our sustainability conditions were derived for a monopoly, they appear to be relevant for the sustainability of other forms of market structure, i.e., perfect competition and oligopoly.[15)]

The conditions for sustainability have important implications for the analysis and measurement of market power. First, a monopolist, even when facing inelastic demand, enjoys a very limited price-making power. The monopolist can exercise his limited price-making power only by choosing a price vector from the sharply narrowed-down set of sustainable price vectors which correspond to a portion of a particular (zero-profit) isoprofit locus. Thus the traditional view of the monopolist as a pricemaker is questionable.[16)] Second, sustainability requires zero economic profit and, therefore, attempts to identify market power with excess economic profits are unjustified. Third, the estimation of the Lerner index and the Bain index is commonly done utilizing direct cost (rents associated with barriers to entry are not included) rather than economic cost. Our sustainability conditions impose restrictions on the relationships between demand and supply conditions, where the latter reflect economic cost relations. Hence, the interpretation of the findings associated with the Lerner and Bain measures depends critically on the shape of the entry cost function which relates direct

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15) We are in the process of completing a paper on this subject.

16) Actually, in the single product setting, the monopolist who is facing decreasing average cost has no price-making power at all. The single sustainable point is defined by the price-output combination which corresponds to zero profit.

cost to economic cost. Unfortunately, very little is known about the shape of the entry cost function or about its determinants.

Figure 2

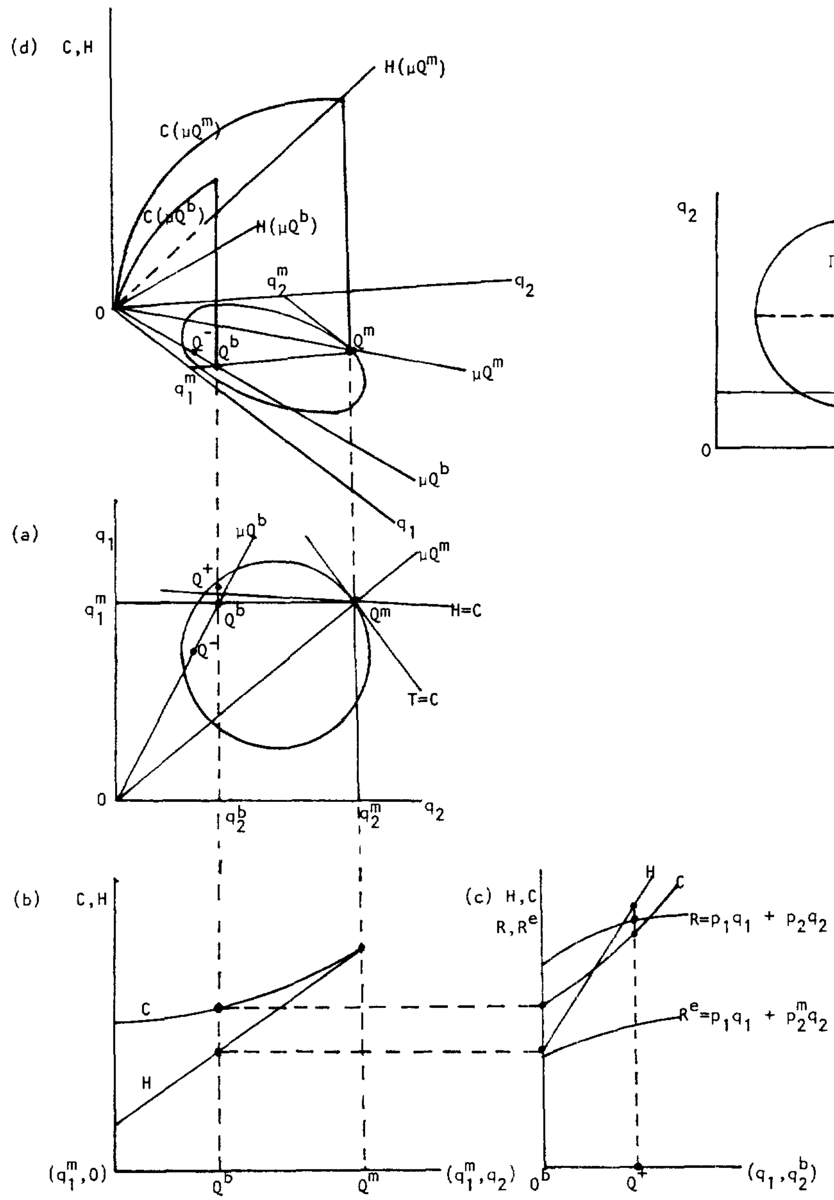
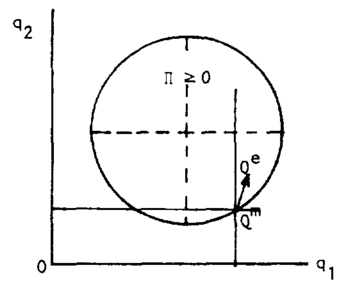


Figure 1



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