A Survey of Quadratic Programming Applications to Business and Economics*

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I. Introduction

Linear programming has been successfully applied in managerial and economic planning. However, it is not without inherent limitations, the most significant being its *linearity* assumption. The linear objective function requires the use of constant returns to scale, constant marginal return, fixed input and output prices, etc.

When increasing or decreasing returns to scale are present in the objective function and/or in the constraints, we must rely on nonlinear programming methods. Unfortunately, nonlinear programming (NLP) has limited practical application.[1] It has typically not been used in solving corporate and economic problems mainly because of the need for large scale computing facilities and various computional difficulties.[2]

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1. A limited number of use of general NLP for economic planning is reported in [9].
2. Efficient, robust and reliable algorithms now exist capable of solving problems with up to 100 variables with higher than quadratic power. These are, however, mainly based on the recursive quadratic programming principle. For these algorithms and their software, see Bartholomew-Biggs and Powell [1] and Waren and Lasdon [24].

Quadratic programming (QP), with *linear* inequality constraints, however, has been widely received by corporate and economic model builders. Several reasons exist for this trend. First, of the nonlinear programming problems, the quadratic formulation is the easiest to solve when one deals with a QP problem involving linear inequality constraints. Indeed, it is only slightly more complex than the linear programming problem. Second, it is easier to handle further mathematical analyses (such as dual variables and sensitivity analysis) by using the quadratic form rather than alternative nonlinear forms. Third, quadratic objective functions provide valid approximations to many preference (utility) functions that assume a zero value at a particular point. Finally, many economic and business problems fit right into the QP formulation, as illustrated here.

The objective of this paper is to present a survey of existing QP practices in corporate and economic planning formulations. In so doing, applications and problem areas are reviewed and illustrated. This survey, however, should not be construed as exhaustive. It is hoped that it will contribute to the wider application of QP in solving real life resource allocation problems.

Classic Applications.[3]

A classic example of the use of QP is the monopolist's profit maximization. Assuming a linear demand function, the profit function will be:

$$\text{Max } \pi = cx - 1/2x'Dx$$

with constraints

$$Ax \leqq b$$
$$x \geqq 0$$

where c is a n vector, D a symmetric (n x n) matrix, b an m vector, and A a (m x n) matrix. This is a typical QP formulation. The traditional problem of diminishing (or increasing) returns to scale can also fit into this formulation. Recently, Hartley [4] and Jensen [6] extended this to the joint cost problem to determine optimal price and output policies.

Another classic application of QP is in portfolio selection centering around what is known as mean-variance analysis. The formulation is:

3. For an excellent discussion of these classics, see McCarl, *et. al.* [14].

$$\text{Max } rx - \lambda x'Vx$$

subject to $xe = 1$

$$x \geqq 0$$

Where x is the proportion of the investor's total investment, r is the expected future return, λ is a coefficient of risk aversion, V the variance or covariance of any given portfolio x and e is the unit vector.

II. Inequality Constrained Least-Squares Estimation

There is an increasing demand to use prior and sample information for parameter estimation in the regression model to achieve consistency with economic theory. For example, the neoclassical demand equation requires the homogeneity condition with properly signed parameter values. To meet such demands, inequality constraint least-squares (ICLS) estimation was introduced by Judge and Takayama [8] and further refined by Liew [11] and Liew and Shim [12].

Consider the following model and *a priori* belief:

$$y = X\beta + u$$
$$A\beta \geqq c\ ,$$

Where X has a full rank (n x k) fixed matrix, A is (m x k) matrix, and y, β, c, and u are n, k, m, n, component vectors. The residual vector u is assumed to satisfy the Gauss-Markov conditions.

Assume b is an estimate vector of B. Then the estimation problem is:

$$\text{Min}_{b}\ Z = 1/2\,(y - Xb)'(y - Xb)$$

$$\text{Subject to } Ab \geqq c$$

Solving this problem, we obtain the ICLS estimate b*:

$$b* = (X'X)^{-1}X'y + (X'X)^{-1}A'\lambda*$$

where λ* is the nonnegative complementary solution of the Dantzig-Cottle fundamental problem. The estimated variance-covariance matrix b* is:

$$\hat{V}(b*) = \frac{u'u}{n-k}\,M(X'X)^{-1}M' \text{ where M is a weighted matrix.}$$

The ICLS method has a practical interest since many empirical studies require inequality restriction, and can be calculated quickly with Dantzig-Cottle algorithms, a property not shared by most Bayesian procedures.

An example[4]:

Suppose we have the following model for the new demand for electricity in the United States.

$$\log q_i = \log \beta_0 + \beta_1 \log p_i + \beta_2 \log \pi_i + \beta_3 \log y_i$$

where q_i is the new demand for electricity in kilowatt hours (KWH) for ith state in 1970, P_i is the price of electricity ($) per KWH for ith state, πi is the price of natural gas ($) per Therm for ith state, y_i is the expenditure on new demands for electricity and natural gas, i.e., $y_i = pi\, qi + \pi i\, z i$, where z_i is the new demand for natural gas for ith state.

To maintain structural consistency, the following restrictions were made:

1. $\beta_1 \leq 0$ (negative price elasticity),
2. $\beta_2 \geq 0$ (positive cross elasticity),
3. $\beta_3 \geq 0$ (positive income elasticity),
4. $\beta_1 + \beta_2 + \beta_3 = 0$ (the homogeneity condition).

The A matrix and c vector in this case become

$$\underset{A}{\begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 \end{bmatrix}} \underset{\beta}{\begin{bmatrix} \log \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}} \geq \underset{c}{\begin{bmatrix} 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \end{bmatrix}}$$

The estimates obtained by the two methods (ICLS vs. Ordinary Least-Squares (OLS)) are shown in Table 1.

The estimated variance-covariance matrix of the ICLS estimates:

$$\hat{V}(b^*) = \begin{bmatrix} 0.0202 & 0.0043 & -0.0025 & -0.0017 \\ 0.0043 & 0.0017 & -0.0015 & -0.0002 \\ -0.0025 & -0.0015 & 0.0015 & -0.0001 \\ -0.0017 & -0.0002 & -0.0001 & 0.0002 \end{bmatrix}$$

4. This example is adapted from [11] and solved by using the ICLS computer program in [12].

TABLE 1

The Estimated Equations

(ICLS vs. OLS)

Independent Variables	ICLS		OLS	
	Estimate	Standard error	Estimate	Standard error
log β_0	-0.2734	0.1421	-0.4229	0.2549
β_1	-1.1734	0.0413	-1.2075	0.0636
β_2	0.2253	0.0391	0.2081	0.0459
β_3	0.9481	0.0140	0.9444	0.0149
R^2	0.9898		0.9899	
Standard error of regression	0.1148		0.1138	

Number of observations = 51

Other examples

Other simple applications of ICLS have been found in the estimation of transition probabilities [21] where the B_i's must be non-negative and in the estimation of Cobb-Douglas type production functions [7] whose parameter values must sum up to either one, greater or less than one. Quite recently, Shim [16], in a paper on cash budgeting, empirically estimated cash collection percentage rates based on credit sales of prior periods. Due to the problem of multi--collinearity in credit sales data the OLS estimate yielded collection rates which were *not* consistent with normal expectations (i.e., negative collection rates, collection rates that increase with the passage of time, etc.). He proposed the use of the ICLS method in cases where (i) a significant problem of multicollinearity exists in the credit sales data causing a negative sign for one or more of the cash collection percentages and/or (ii) a prior expectation requires other inequality restrictions on the collection pattern.

III. Goal Programming with Quadratic Preferences

Goal programming (GP) was first developed by Charnes and Cooper [2], and further refined by many authors (for example, [10, 18]). This method is being well received by model builders because it incorporates multiple and conflicting goals. However, one of its shortcomings lies in the *linearity* assumption, especially in the objective function. Hence, we are compelled to work with a *constant* marginal rate of substitution between goals. Recently, Shim and Siegel [17] incorporated quadratic preferences into the GP structure in an effort to conform to the economist's utility postulates. The formulation involves finding vectors x and d such that:

$$\text{Max } U(d) = -wd'Qd$$

subject to

$$Ax + Ed = b$$

$$x, d \geqq 0$$

In this formulation, A is a (m x n) matrix of coefficients, x is a (n x 1) vector of decision variables, E is a (2m x m) matrix, and b is a (m x 1) vector of desired goal levels. d, (d^-, d^+), is a (2m x 1) vector representing negative and positive deviations from target goals. w, (w^-, w^+), is a (1 x 2m) vector of weighted priority factors.

An Example:

A textile company... produces two types of linen materials—a strong upholstery material and a regular dress material. The upholstery material is produced according to direct orders from furniture manufacturers. The dress material, on the other hand, is distributed to retail fabric stores. The average production rates for the upholstery material and for the dress material are identical: 1000 yards per hour. By running two shifts, the operational capacity of the plant is 80 hours per week.

The marketing department reports that the maximum estimated sales for the following week is 70,000 yards of the upholstery material and 45,000 yards of the dress material. According to the accounting department, the approximate profit from a yard of upholstery material is \$2.50, and from a yard of dress material \$1.50...

The president of the company believes that a good employer-employee relationship is an important factor for business success. Hence, he decides that a stable employment level is a primary goal for the firm. Therefore, whenever there is demand exceeding normal production capacity, he simply expands production capacity by providing overtime. However, he also feels that overtime operation of the plant of more than 10 hours per week should be avoided because of rising costs. The president has the following four goals:

1. To avoid any underutilization of production capacity.
2. To limit the overtime operation of the plant to 10 hours.
3. To achieve the sales goals of 70,000 yards of upholstery material and 45,000 yards of dress material.
4. To minimize the overtime operation of the plant.[5]

The goal programming model (called hereafter Case A) of this example is as follows:

$$\text{Min } U(d) = p_1 d_1^- + p_2 d_4^+ + 5p_3 d_2^- + 3p_3 d_3^- + p_4 d_1^+$$

$$\begin{aligned}
\text{s.t.} \quad & x_1 + x_2 + d_1^- - d_1^+ = 80 \\
& x_1 + d_2^- = 70 \\
& x_2 + d_3^- = 45 \\
& x_1 + x_2 + d_4^- - d_4^+ = 90 \\
& x_1, x_2, d_1^-, d_1^+, d_2^-, d_3^-, d_4^-, d_4^-, d_4^+, \geq 0
\end{aligned}$$

where x_1 = hours of production of upholstery material

x_2 = hours of production of dress material

d^-_1 = underutilization of production capacity as set at 80 hours/week

d^+_1 = overutilization of production capacity as set at 80 hours/week

d^-_2 = underachievement of sales goal for upholstery material

d^-_3 = underachievement of sales goal for dress material

d^-_4 = negative deviation of overtime operation from 10 hours overtime

d^+_4 = overtime beyond 10 hours

5. This example is taken from [10].

Incorporating quadric preferences, the formulation (called hereafter Case B) is:

$$\text{Min } U(d) = p_1 (d_1^-)^2 + p_2 (d_4^+)^2 + p_3 [(5d_2^-)^2 + (3d_3^-)^2] + p_4 (d_1^+)^2$$

Subject to the constraint set in Case A.

The solution to this problem is compared with the one by linear GP, in Table 2. The solution occurs at point B as shown in Figure 1, while the solution to linear GP corresponds to point A. As expected the quadratic formulation resulted in a willingness to a accept a larger deviation of the sales goal for upholstery material (d^-_2) in order to satisfy more of the sales goal for lining material. The marginal rate of substitution of d^-_2 for d^-_3 has changed from a constant ratio of 5 to 3 in the linear formulation to $5d^-_2$ to $3d^-_3$ in the quadratic formulation.

Note that as Shim and Sigel indicated, interaction terms may also be introduced into quadratic goal programing formulations to reflect the asymmetrical nature of the decision maker's preference function.

Table 2

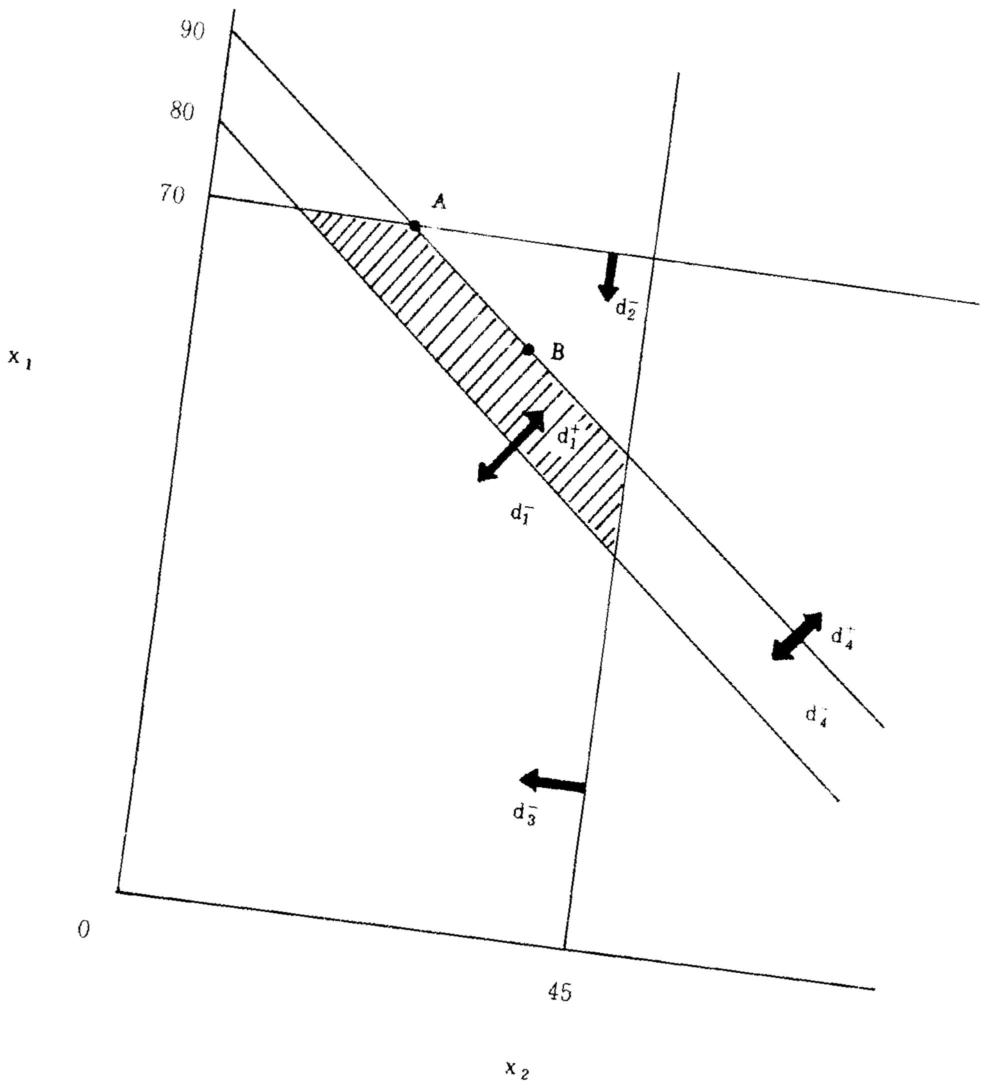
Optimal Solutions

(GP vs Quadratic GP)

Goal Programming	Quadratic Goal Programming
$x_1^* = 70$	$x_1^* = 60.625$
$x_2^* = 20$	$x_2^* = 29.375$
$d_1^{-*} = 0$	$d_1^{-*} = 0$
$d_1^{+*} = 10$	$d_1^{+*} = 10$
$d_2^{-*} = 0$	$d_2^{+*} = 9.375$
$d_3^{-*} = 25$	$d_3^{-*} = 15.625$
$d_4^{-*} = 0$	$d_4^{-*} = 0$
$d_4^{+*} = 0$	$d_4^{+*} = 0$
$U(d^*) = 75P_3 + 10P_4$	$U(d^*) = 1171.875\,P_3 + 100\,P_4$

FIGURE 1

Graphical Illustration



IV. Spatial Equilibrium Analysis for Optimal Allocation and Pricing

Spatial equlibrium analysis primarily involves the determination of inter-regional allocation and pricing patterns for a commodity. This approach has been researched extensively by Takayama and Judge [19] and Liew and Shim [13], and has many economic applications (for example, see [15, 3]).

Spatial equilibrium analysis prima ily relates to the formulation of a net quasi-welfare (NW) function based on *linear* regional demand and supply functions, which is expressed in quadratic form.[6]

Consider an economy with n regions and m homogeneous commodities. It is assumed that the demand and the supply equations are linear functions of prices.

$$d = \alpha + D \cdot P^d,$$
$$S = \theta + S \cdot P^S. \qquad (1)$$

Where d and s are properly stacked vectors of quantities demanded and supplied. α and 0 are parameter vectors. The behavioral coefficient matrices D and (−S) are assumed to be symmetric and negative definite. P^d and P^S are properly stacked vectors of demand and supply prices.

To simplify the presentation, we define;

$$c \equiv \begin{bmatrix} \alpha \\ \cdots \\ -\theta \end{bmatrix} \quad Q \equiv \begin{bmatrix} D & 0 \\ 0 & -S \end{bmatrix} \quad P \equiv \begin{bmatrix} p^d \\ \cdots \\ p^s \end{bmatrix} \quad y \equiv \begin{bmatrix} d \\ \cdots \\ s \end{bmatrix} \qquad (2)$$

Let x and t be the properly stacked commodity flow vectors and the corresponding unit transportation cost vectors.

The net quasi-welfare function of the economy can be formulated in terms of either the price vector or the quantity and commodity flow vectors. Only price formulation is presented in this paper (for a quantity formulation, see [13, 19]).

Let ξ (p) be the vector function of demand and supply relations with negative sign on

6. It should be noted that the spatial equilibrium model becomes the classical Hitchcock ***transportaion cost minimization*** problem when regional demand and supply are not dependent on price.

supply relations.[7] The usual formulation of net welfare function involves a line integral of the vector function ξ (p) in terms of the price vector along A which is a set of points between the pre-trading prices and the post-trading prices; i.e.,

$$\underset{p}{\text{Max}}\ NW = \int_A \xi(p)\cdot dp \equiv p'c + \tfrac{1}{2}p'Qp$$

subject to

$$Gp \leqq t,$$

$$p \geqq 0.$$

Where G is a restriction matrix such that the demand price of the kth commodity in region j (P_{jk}^{d}) should be not greater than the sum of the transportation costs (t_{ij}^{k}) required to deliver the kth commodity from region i to j and supply price of kth commodity in region i (p_{ik}^{s}), i.e., $\{GP \leqq t\}$ is a matrix form of $\{p_{jk}{}^{d} - p_{ik}{}^{s} \leqq t_{ij}^{k}$

where i, j = 1, ... , k = 1, . . . , m }

<u>An Example:</u>

Consider the following system of demand and supply equations for an economy which has two regions and two commodities;

$$\begin{bmatrix} d_{11} \\ d_{12} \\ d_{21} \\ d_{22} \end{bmatrix} = \begin{bmatrix} 200 \\ 300 \\ 100 \\ 150 \end{bmatrix} + \begin{bmatrix} -10 & 1 & 0 & 0 \\ 1 & -10 & 0 & 0 \\ 0 & 0 & -5 & 1 \\ 0 & 0 & 1 & -5 \end{bmatrix} \begin{bmatrix} p_{11}^{d} \\ P_{12}^{d} \\ P_{21}^{d} \\ p_{22}^{d} \end{bmatrix}$$

or

$$d = \alpha + D\cdot pd$$

7. From (1) and (2), $\begin{pmatrix} d \\ \cdots \\ -s \end{pmatrix} = c + Qp \equiv \xi(p).$

$$\begin{bmatrix} s_{11} \\ s_{12} \\ \\ s_{21} \\ s_{22} \end{bmatrix} = \begin{bmatrix} -50 \\ -60 \\ \\ 15 \\ -60 \end{bmatrix} + \begin{bmatrix} 10 & 0.5 & 0 & 0 \\ 0.5 & 15 & 0 & 0 \\ \\ 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0.5 & 25 \end{bmatrix} \begin{bmatrix} p_{11}^s \\ p_{12}^s \\ \\ p_{21}^s \\ p_{22}^s \end{bmatrix}$$

or

$$s = \theta + S p^S.$$

This function is one cornerstone since it is the constraint of the decision maker. For the other cornerstone, the preference function, he sets up the quadratic form as follows:

$$Z(x, y) = h(x - x^T)^2 + k(y - y^T)^2$$

where x^T and y^T are desired levels of x and y, and h and k are weights assigned to these goals. This weighted sum of squares, Z (x, y), is to minimized. Carrying out this conditional minimization problem is an easy task since the constraint is basically equality. The result is:

$$x^0 - x^T = \frac{hb}{h + kb^2} (y^T - a - bx^T)$$

where the left hand side contains the optimal decision x^0, measured as a deviation from the desired level of government expenditure x^T.

Holt and other [5] took a similar position in finding an optimal rule for employment and production decisions. Their work deals with cost minimization in a paint factory based on using a quadratic cost function with equalities. Numerical derivations of linear decision rules can be found in many texts such as [22, 23].

VI. Summary and Conclusion

A survey has been presented on the use of quadratic programming in solving business and economic planning problems. It is *by no means* exhaustive and conclusive. For example, as McCarl, *et. al.* [14] noted, quadratic functions (in the form of second order Taylor series expansions) have long been used to approximate nonlinear functions. Nevertheless, it represents a review of the practical applications of quadratic programming specifically to business and economics. The optimization model discussed here is more consistent with the decision maker's preference postulates than is the linear programming model. Furthermore, QP formulations are more realistic than is LP in numerous business applications relating to the management decision making process.

We wish to maximize the net welfare function in terms of prices with the following restrictions:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} p_{11}^d \\ p_{12}^d \\ p_{21}^d \\ p_{22}^d \\ p_{11}^s \\ p_{12}^s \\ p_{21}^s \\ p_{22}^s \end{bmatrix} \leqq \begin{bmatrix} 0 \\ 2 \\ 2 \\ 0 \\ 0 \\ 3 \\ 3 \\ 0 \end{bmatrix}$$

or

$$Gp \leqq t.$$

The following optimal solutions from the price formulation were obtained:

$$\begin{aligned} p_{11}^d &= p_{11}^s = 12.84 & \lambda_1 &= 83.88 = x_{11}^1 & \lambda_5 &= 130.19 = x_{11}^2 \\ p_{12}^d &= p_{12}^s = 12.25 & \lambda_2 &= 0.61 = x_{12}^1 & \lambda_6 &= 0.00 = x_{12}^2 \\ p_{21}^d &= p_{21}^s = 14.84 & \lambda_3 &= 0.00 = x_{21}^1 & \lambda_7 &= 60.13 = x_{21}^2 \\ p_{22}^d &= p_{22}^s = 9.25 & \lambda_4 &= 34.46 = x_{12}^1 & \lambda_8 &= 118.58 = x_{24}^2 \end{aligned}$$

where λ is the dual vector of the price formulation which, as proved in [13], is the same as the net welfare maximizing commodity flow vector (x).

V. Optimal Linear Decision Rules

Theil [20] proposed the use of quadratic utility and loss functions in business and government decision making and also showed how to derive a simple ***linear decision rule.*** To illustrate, he let the instrument variable x be governmental expenditure and the non-controlled variable y be GNP. He then let GNP be a linear function of government expenditure:

$$y = a + bx$$

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