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A Multistage Mathematical Programming Model for Aggregate Production Planning and Operations

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Sophisticated general purpose mathematical programming systems are available from numerous sources, both academic and commercial. This paper documents the application of one of the most widely used commercial systems, IBM's Mathematical Programming System-Extended(MPSX). A general MPSX-multistage algorithm is developed and applied to the solution and simulated usage of a linear programming aggregate operations planning model. Deterministic applications of a multiperiod linear programming aggregate operations model are simulated in exploring the sensitivity of solutions to changes in both the horizon length and ending conditions. The results of these several different deterministic applications are briefly discussed in the paper.

I. INTRODUCTION

Sophisticated general purpose mathematical programming systems are available from numerous sources. Two typical systems are IBM's Mathematical Programming System - Extended (MPSX) [9] and Northwestern University's Mathematical Programming Optimization System (MPOS) [2]. These systems have evolved as substantial improvements have been made in both computer technology and mathematical pro-

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gramming codes. Both of these systems integrate significant data handling capabilities with their efficient mathematical algorithms. Such data handling capabilities very often make the implementation of Operations Research/Management Science (OR/MS) models more economical in a production environment.

This paper discusses the application of IBM's MPSX in solving and simulating the usage of a linear programming model for aggregate production planning and operations. An algorithm is developed through the use of an MPSX control language program and an external FORTRAN program. This algorithm is used to simulate the many recursive applications of two different linear programming models. Before discussing the actual algorithm, the model and problem which prompted the use of MPSX procedures will be developed.

II. THE PROBLEM

Optimization techniques of OR/MS have been combined with both simplified and sophisticated models. A major effort of OR/MS study has been directed to the derivation of methods which would optimize the operations of corporate systems. One set of methods of optimization is that subset known as "aggregate operations planning models." These models are also known as aggregate production planning and scheduling models, production smoothing models, or operations planning models. These terms will be used interchangeably in this paper.

The aggregate operations planning and scheduling problem has been researched considerably since the development of one of the first techniques-the Linear Decision Rule (LDR). Holt, Modigliani, and Simon [8] developed the LDR in 1955, and since that time many additional models have been formulated as vast improvements have been made in computer-oriented optimization techniques.

These models are aggregate in that individual products, machines, laborers and other specific resources are not scheduled. The costs, benefits, and the mathematical relationships of the specific rescurces available to the decision maker are expressed in a common unit of aggregation. Thus, the parameters and variables used in the specific model are expressed in a unit such as actual or standard machine hours, actual or standard labor hours, or some other unit of analysis. Consequently, the objective of the aggregate planning model becomes one of producing, inventorying, and subcontracting, for example, standard hours of machine time so as to minimize the cost (or maximize the profits) of supplying machine time while facing variations in demand for that product or service.

When applying an aggreate model with a finite horizon, the long-run optimal solution is that solution which minimizes the long-run costs of meeting current and future product demands. Intuitively one feels that the determination of the current month's optimal operating decisions does not usually require the solution of the long-run or infinite-horizon problem. In fact it has been shown the horizon which influences a current month's decisions very often does not extend more than a single seasonal cycle into the future, and is often much shorter [15]. The inclusion of future periods outside of this influencing horizon is a needless complexity with respect to the initial (or current) month's decisions.

As the complexity and sophistication of decision models have increased, questions concerning the length of the planning horizon to be included in these models have also increased. As in most economic decisions, the correct choice of a planning horizon involves several trade-offs. In determining a model's horizon length, the decision-maker must make trade-offs between solution optimality, computational complexity, and model realism.

This study was prompted by a previous study involving the formulation and application of a linear programming aggregate operations model of a medium-size plastic part manufacturer [12]. While developing that model it was noted that its linear programming solutions were sensitive to both the length of the planning horizon and the inventory parameters of the last period of that horizon. To improve the actual managerial application of that model, this research was undertaken. Of particular interest was the variation of total costs from the optimum as the length of the planning horizon was varied and as changes were made to the ending inventory parameters of the horizon. This original model is discussed briefly here, for a more detailed description one should consult the original source [12].

III. THE MATHEMATICAL FORMULATION

The Model

The objective of the model is the determination of the optimal monthly levels of aggregate regular production (P_i), overtime production (Q_i), the employment level (as determined by the production rate p_i/t_i), and the inventory level (which is determined by I_i^+ or I_i^-). The aggregate unit used in the model is "standard" hours of machine time. Table 1 below lists the notations used in the mathematical formulation of the model.

Table 1
Lits of Notations

Variables	
C	$=$Total cost for the n-month period
P_i, (Q_i)	$=$Number of standard hours produced no regular (overtime) time in period i
ΔP_i^+, (ΔP_i^-)	$=$Increase (decrease) in the production rate in standard hours per day in period i
I_i^+, (I_i^-)	$=$Excess (shortage) in ending inventory in standard hours above I_i (below I_i) in period i
I_i	$=$Ending inventory of period i
Parameters	
C_{i1}, (C_{i2})	$=$Regular (overtime) production labor cost per standard hour in period i
C_{i3}, (C_{i4})	$=$Cost of increasing (decreasing) the average daily production

	rate in any period by one standard hour per day in period i
$C_{i5}, (C_{i6})$	= Inventory cost per standard hour in excess of (below) the upper (lower) limit of optimal inventory level during i
P_0/t_0	= Beginning production rate of period 1
I_0	= Beginning inventory level of period 1
I_i', (I_i'')	= Minimum (Maximum) inventory allowed in period i
$\hat{I}_i, (\check{I}_i)$	= Upper (lower) limit of optimal inventory band in period i
S_i	= Demand for standard hours in period i
t_i, (t_i')	= Number of regular (overtime) production days available in period i
L_i	= Maximum regular production level in standard hours in period i ($L_i = M_i t_i$)
M_i	= Maximum production rate possible in standard hours per day in period i
i	= Current period, $i = 1, 2, \cdots, n$
j	= Time periods, $j = 1, 2, \cdots, i$
n	= Length of the planning horizon in months
N	= The Nth month of the T months
T	= Total number of periods used in simulated usage

The linear programming model is given below.

Minimize:

$$C(P_i, Q_i, \varDelta P_i^+, \varDelta P_i^-, I_i^+, I_i^-) = \sum_{i=1}^{n} C_{i1}P_i + \sum_{i=1}^{n} C_{i2}Q_i + \sum_{i=1}^{n} C_{i3}\varDelta P_i^+$$

$$+ \sum_{i=1}^{n} C_{i4}\varDelta P_i^- + \sum_{i=1}^{n} C_{i5}I_i^+ + \sum_{i=1}^{n} C_{i6}I_i^-$$

subject to:

$$M_i - P_i/t_i \geq 0, \qquad i = 1, 2, \cdots, n \qquad (1)$$

$$P_i/t_i - Q_i/t_i' \geq 0, \qquad i = 1, 2, \cdots, n \qquad (2)$$

$$\varDelta P_i^+ - P_i/t_i + P_{i-1}/t_{i-1} \geq 0, \qquad i = 1, 2, \cdots, n \qquad (3)$$

$$\varDelta P_i^- + P_i/t_i - P_{i-1}/t_{i-1} \geq 0, \qquad i = 1, 2, \cdots, n \qquad (4)$$

$$I_i^+ + \hat{I}_i + \sum_{j=1}^{i} [-I_0 - (P_j + Q_j) + S_j] \geq 0, \qquad i=1,2,\cdots,n \qquad (5)$$

$$I_i^- - \check{I}_i + \sum_{j=1}^{i} [I_0 + (P_j + Q_j) - S_j] \geq 0, \qquad i=1,2,\cdots,n \qquad (6)$$

$$\check{I}_i - I_i^- - I_i^- \geq 0, \qquad i=1,2,\cdots n \qquad (7)$$

$$I_i'' - I_i^+ - \hat{I}_i \geq 0, \qquad i=1,2\cdots,n \qquad (8)$$

and

$$P_i, Q_i, \Delta P_i^+, \Delta P_i^-, I_i^+, I_i^- \geq 0, \qquad i=1,2,\cdots,n \qquad (9)$$

where

$C_{i1}, C_{i2}, C_{i3}, C_{i4}, C_{i5}, C_{i6}, I_0, \hat{I}_i, \check{I}_i, I_i', I_i'', M_i, S_i, t_i$ and t_i' are known constants. (The cost figures used in the model are: $c_{i1}=\$4.53$, $c_{i2}=\$7.80$, $c_{i3}=\$25.80$, $c_{i4}=\$22.08$, $c_{i5}=\$.26$, and $c_{i6}=\$1.20$)

A brief explanation of period i constraints is given below:

Production and Employment Constraints:

(1) limits the maximum allowable production level (P_i) and rate (P_i/t_i) during regular production,

(2) limits the maximum allowable production level (Q_i) and rate (Q_i/t_i') during overtime production,

(3) defines production rate increases (ΔP_i^+),

(4) defines production rate decreases (ΔP_i^-),

Inventory Constraints:

(5) is an interperiod constraint defining the excessive inventory level I_i^+,

(6) is an interperiod constraint defining the level of inventory below the optimum inventory level (I_i^-),

(7) defines the lower bounds on the minimum inventory level (I_i'), and

(8) defines the upper bounds on the maximum inventory level (I_i'').

Optimal Inventory Levels - A Distinction

In discussing optimal levels of aggregate inventory, there is an important distinction that must be made concerning two different concepts. This distinction is between the concepts of "optimal ending inventory" and "long-run optimal ending inventory."

The term "optimal ending inventory" refers to the lot-size determined optimal ending inventory of period i. A basic economic order quantity model was used to determine the optimal production lot size for each piece part. By adding, for each part, the average safety stock to one-half the economic lot size, a given month's optimal average inventory level is determined. Then, by adding together these optimal average inventories for all parts that are stocked, an optimal aggregate average inventory for all the parts is obtained [7].

An optimal average inventory band ($\hat{I}_i - \check{I}_i$) was determined for each month as the variations in cost throughout this band was very small. When inventories fall below the band ($\check{I}_i$ with $I_i^- > 0$), a shortage expense ($c_{i6} = \$1.20$) is incurred. When inventories are in excess of this band ($\hat{I}_i$ with $I_i^+ > 0$) an excess inventory carrying cost ($c_{i5} = \$.26$) is incurred. In determining the ending inventory level of a period, it is assumed that the inventory level at the end of each month is the average inventory level during the month.

In contrast to the phrase "optimal ending inventory level," the use of the phrase "long-run optimal ending inventory level" refers to that ending inventory of a period which is necessary to minimize the T-period costs of allocating aggregate resources. The "long-run optimal ending inventory" levels are determined by solving the total aggregate model, not through the single application of a lot-size formula. The distinction between the two concepts is very important one.

Model Assumptions

In formulating the model, the following assumptions were found to be valid:

1. Demand for production is deterministic and given in standard hours as S_i in period i $(i=1, 2, \cdots, n)$.
2. The variable cost of production in period i is a piecewise linear convex function.
3. The cost of changes in the production rate $(\Delta P_i^+, \Delta P_i^-)$ during period i is a piecewise linear function.
4. The cost of changes in the production rate $(\Delta P = P_i/t_i - P_{i-1}/t_{i-1})$ is independent of the previous production rate (P_{i-1}/t_{i-1}).
5. The total inventory cost function for all levels of demand S_i is a common piecewise linear function.
6. The regular production rate is limited to 148 standard hours per day. Additional or decreased capacity through changes in the number of machines is not a decision variable.
7. A single production facility is assumed; the facility produces a single product.
8. Back-orders are not explicitly allowed in the model.
9. Material costs and other indirect expenses are fixed within the region of study.
10. The model assumes a three-shift operation without the alternative of starting-up and shutting-down a given shift (i.e., no shift set-up costs).
11. Production scheduled for the month is produced within the month and available for supplying demand occurring during the month.

IV. THE SIMULATION PROCEDURE

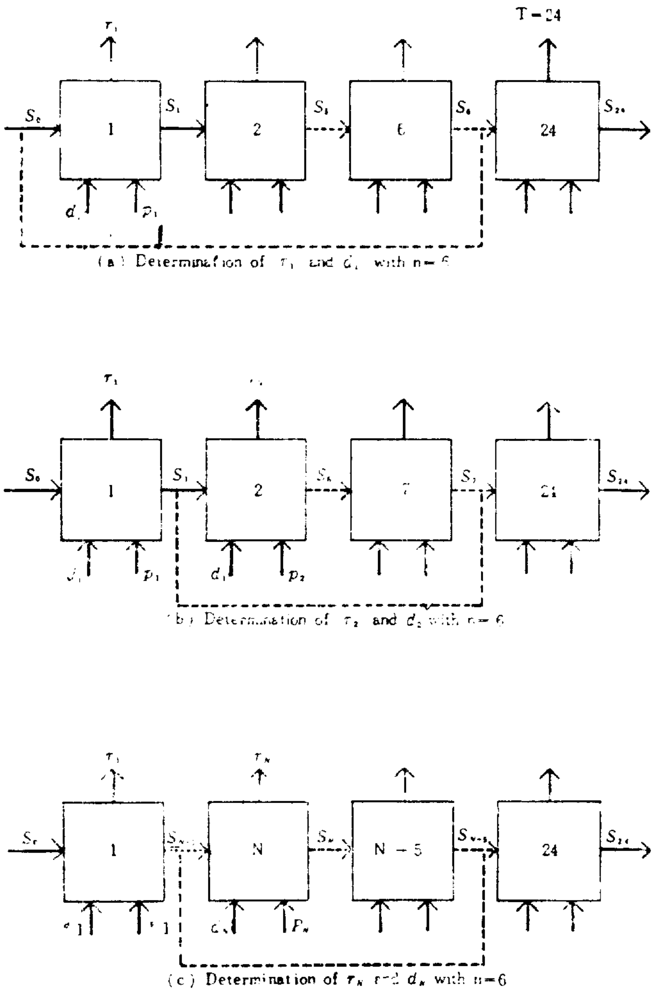
The Linear Programming (LP) Algorithm

The simulation of many different deterministic applications of a linear programming model required an efficient computer algorithm. Several thousand separate linear programming problems were solved in these simulations. An MPSX algorithm was developed to simulate the usage of an n-period model over T periods. This algorithm is described below.

Using actual past demands and constraints facing a manufacturer, 24 months of applying the basic model and modifications to it are simulated. In simulating 24 months of using the model, different n-month models (e.g., n=6 months) are applied 24 times to arrive at the total cost of using the first-month decisions of each of the 24 applications. The total costs are then compared to determine the differences in costs resulting from variations in the horizon lengths and/or differences in monthly ending conditions. The costs included in the total costs are the costs which were incurred in the first month of each of the T applications of n-month model. This procedure is best described while viewing Figure 1 which uses general multistage representation and notations.

Figure 1 (a) depicts the first n-month solution (i.e., n=6) of T solutions (i.e., T=24). The cost or return (r_1) incurred in the first month is determined by solving the six-month linear programming model for the optimal decisions. (The solution is optimal only with respect to the six months included in the model, and not necessarily optimal with respect to the 24-month period or the long run.) After the initial six-month solution, it is assumed that the first month's decisions (d_1)--i.e., P_i and Q_i--are followed. These decisions result in the incurrence of first month cost as well as the establishment of the starting conditions (s_1)--i.e., beginning inventory level (I_1) and the beginning production rate (P_1/t_1)--of the second month. The first month cost (r_1) just determined is the first cf the costs which will be summed for making comparisons.

The first month's decisions (d_1) arrived at in this solution are partially dependent upon the input state vector (s_0)--i.e., I_0 and P_0/t_0-- and month six's output state vector (s_6). The components of the input state vector for period i are the values of the beginning inventory level (I_{i-1}, as determined by I_{i-1}^{+} or I_{i-1}^{-}) and the beginning production rate (P_{i-1}/t_{i-1}). While the input state vector of month one (s_0) is known with certainty, the optimal value of the output state



Multistage Notations:

r_i = returns (costs) incurred in period i

s_i = output (input) state vector of period i (i+1); i.e., ending (beginning) production rate, P_i/t_i and ending (beginning) inventory level, I_i

d_i = decisions of period i; i.e., P_i and Q_i

P_i = parameters; i.e., linear programming coefficients and constants

Figure 1. The Multistage Simulation Procedure

vector for month six (s_6) is not known.

The output state vector of the last month of a horizon is partially determined by the inventory and producion parameters of the last month. This research was initiated to study the total cost effects resulting from different modifications to the inventory parameters of the last month of a horizon (i.e., I_n',I_n'',C_{n5}, and C_{n6}).

Having established the input state vector for period two (s_1)--i.e., I_1 and P_1/t_1--an additional month is added to the planning horizon. A six-month model running from months two through seven is now the decision model of interest. Figure 1 (b) displays the next step in the simulation, which is the determination of the second period cost (r_2) and decisions (d_2).

This six-month model is solved as the previous six-month model. Its solution results in the incurrence of period-two cost (r_2) as well as the establishment of the input state vector of period three (s_2). The cost for period two is recorded as was that of period one. This procedure is repeated until the summation of T-period costs is complete. This is calculated as shown below:

$$\text{Total Cost} = \sum_{i=1}^{T} r_i$$

As a general example of the simulation procedure, the determination of r_N and d_N is illustrated in Figure 1 (c). The example given in Figure 1 using an n of six months and a T of 24 months is only one of the many different combinations of models and horizons lengths employed in this study.

In summary, we see that the total cost is the summation of the T costs incurred in the first month of each of the n-month horizons. The procedure just described simulates the managerial use of a model under deterministic conditions. In applying the model, the procedure simulates a manager who makes monthly decisions using a finite-horizon model to determine optimal decisions. For this and other past studies [8, 17, 20] the simulated total cost of using the model under deterministic conditions is the measure used in distinguishing one model as being more valid than another.

Having completed a general discussion of the simulation algorithm of this research, a brief description and discussion of its implementation using MPSX and FORTRAN as computational languages is given in the following section.

V. MPSX PROCEDURES

MPSX is a general mathematical programming (MP) system having the capability of solving linear and linear separable programming (LP), mixed integer programming (MIP) and generalized upper bounding (GUB) problems. "The linear programming procedures of MPSX use the bounded variable/product form of the inverse/revised simplex method" [9]. MPSX has many advantages, one of which is the capability to solve extremely large problems. Depending upon the particular installation, MPSX can solve problems with somewhat greater than 16,000 rows.

In solving the mathematical programming problems listed above, the user of MPSX has a surprisingly large number of programming procedures at his disposal. This research used several of these procedures. These procedures will be described briefly when possible.

The solution of MP problems through MPSX is completed through the ordered execution of a series of procedures much as one would program any other high-level language such as FORTRAN. These procedures are called "Procedure Call Statements" and are used in a "Control Language Program."

MPSX procedures provide the user with substantial flexibility in programming MP computations, solutions, and simulations. In addition to internal procedures, MPSX allows the user to call external FORTRAN, PLI, and COBOL programs to read and manipulate MPSX inputs, procedures, and outputs. The capability of calling a FORTRAN procedure made the computerization of the LP algorithm of the previous section relatively easy to implement through MPSX.

Three specific modules are used in simulating the managerial use of the LP model. These three modules were the MPSX Input Data File, MPSX Control Language Program, and an external FORTRAN program. The Control Language Program is the master program, it contains the logic of the algorithm in its procedure call statements. It controls both the external FORTRAN program as well as the revisions and solutions of the Input Data File.

The MPSX Input Data File contains the LP matrix of the multi-period model (e.g., a six-month model). It is this basic data file which contains the coefficients and right-hand-side constants (i.e., the parameters P_i) of every month of the model.

Figure 2 provides an information flow diagram of the procedure described previously in Section IV, while Figure 3 provides a partial MPSX listing of the Control Language Program. Figure 3 also provides comment statements to introduce and explain the MPSX procedure call statements which are all indented. The explanations of the procedure call statements precede each statement and are denoted with double asterisks. A brief explanation of Figures 2 and 3 will now be given.

Stage one of both Figures 2 and 3 represents the solution of an initial n-month model (i.e., n=6). The first FLAGS statement of Figure 3 defines the horizon length, which is the first six months of the Input Data File "PRODMO". It is this subset of the original Input Data File which is solved. After the PRIMAL solution, the solution basis just completed is SAVEd as a starting basis for the next n-month model. MPSX then outputs the values of the decision variables (i.e., d_1) of the first n-months through SOLUTION to a user defined FORTRAN file. This file is then read by the external FORTRAN program SAD1.

SAD1 outputs to another file a REVISEd set of parameters for the next planning horizon. These parameters are the vectors which "freeze" (through an equality constraint) the first-month decision variables (i.e., it establishes s_1, r_1, d_1, and p_1) to values arrived at in the first

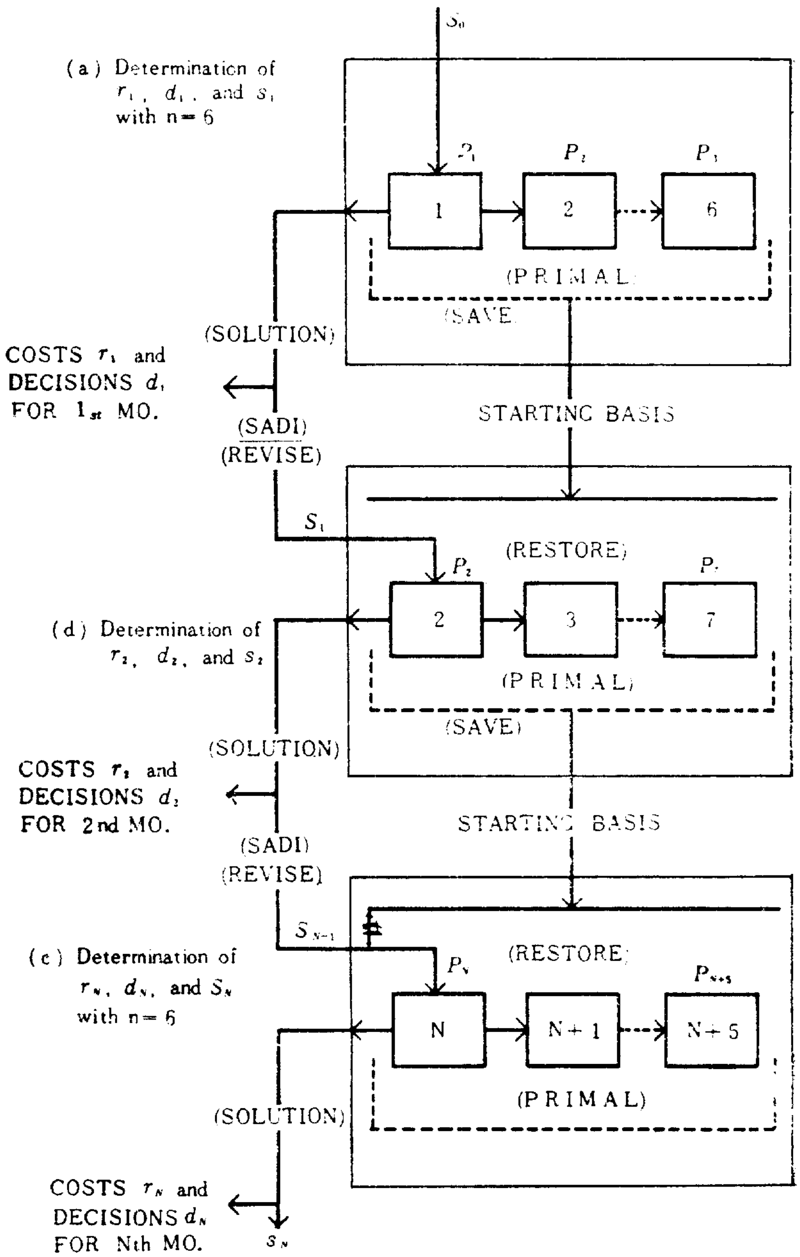


Figure 2. MPSX Information Flow Diagram
(MPSX Procedure Call Statements in Parenthese)

```
**STAGE ONE**
**DEFINES USER INPUT DATA FILE (MATRIX) NAME**
          MOVE (XDATA, 'PRODMO')
**DEFINES USER PROBLEM NAME**
          MOVE (XPBNAME, 'PBFILEI')
**DEFINES A PARTICULAR RHS**
          MOVE(XRHS, 'CON')
**DEFINES A PARTICULAR OBJECTIVE FUNCTION**
          MOVE(XOBJ, 'TOTOCOST')
**CONVERTS PROBLEM INTO BINARY FORMAT**
          CONVERT
**ORGANIZES THE PROBLEM FOR MINIMIZATION**
          SETUP
**DEFINES A SUBSET OF ORIGINAL PROBLEM(DEFINES THE
**HORIZON LENGTH)**
          FLAGS('RLIMIT', 'M7', ' ', 'CLIMIT', 'REG7',' ')
**SOLVES THIS PROBLEM SUBSET**
          PRIMAL
**SAVES THIS SOLUTION AS AN INITIAL SOLUTION FOR
**A FUTURE PROBLEM**
          SAVE ('NAME', 'BASIS')
**OUTPUTS THE SOLUTION TO A FILE**
          SOLUTION('FILE', 'FT04F001')
**CLOSES THE FILED SOLUTION**
          CLOSEF('FT04F001')
**CALLS A FORTRAN PROGRAM TO READ FT04F001**
**THIS PROGRAM ALSO OUTPUTS TO FILE FT09F001 THE
**'REVISE' STATEMENT FOR MONTH 1**
          SAD1
**STAGE TWO**
**DEFINES A NEW USER INPUT DATA FILE NAME THAT
**APPEARS ON FT09F001**
          MOVE(XDATA,'PRODM1')
**DEFINES A NEW PROBLEM NAME**
```

Figure 3. MPSX Control Language Program

```
            MOVE(XPBNAME, 'PBFILE2')
**DEFINES THE OLD PROBLEM NAME**
            MOVE(XOLDNAME, 'PBFILE1')
**REVISES OLD PROBLEM TO NEW PROBLEM PER DATA OF
**FT09F001 WHICH FREEZES FIRST MONTH DECISION VAR-
**IABLES TO VALUES OF THE FIRST SOLUTION**
            REVISE('FILE', 'FT09F001')
            SETUP
            FLAGS('RLIMIT', 'M8', ' ', 'CLIMIT', 'REG8', ' ')
**RESTORES PREVIOUS PROBLEM SOLUTION AS THE FIRST
**FEASIBLE SOLUTION ITERATION**
            RESTORE('NAME', 'BASIS')
            PRIMAL
            SAVE('NAME', 'BASIS')
            CLOSEF('FT04F001')
            SAD1
            *
            *
            *
 **STAGE T**
            MOVE(XDATA, 'PRODM(M)')
            MOVE(XPBNAME, 'PBFILE(M)')
            MOVE(XOLDNAME, 'PBFILE(M-1)')
            REVISE('FILE', 'FT09F001')
            SETUP
            FLAGS('RLIMIT', 'M(M+6)', ' ', 'CLIMIT',
              'REGM(M+6)',' ')
            RESTORE
            PRIMAL
**PRINTS OUT FINAL SOLUTION**
            SOLUTION
**RETURNS CONTROL TO OS/360**
            EXIT
**DENOTES END OF CONTROL PROGRAM**
            PEND
```

Figure 3. MPSX Control Language Program(Continued)

solution. This "freezing" occurs by modifying and updating the coefficients and constants of the first month of the planning horizon. The proccedure repeats itself with the exception that prior to the PRIMAL solution, MPSX RESTOREs the previous problem basis as the first solution basis.

The procedure outlined above was repeated in the Control Language Program through multiple statements, it can also be controlled through a "looping" procedure. The number and versatility of MPSX procedure call statements equips the user with a large number of different approaches to the same problem.

VI. SIMULATION RESULTS

Table 2 presents the results of 24 months of the simulated usage of both the original model formulated in Section III anda single modification of that model. The model was modified by eliminating the shortage strategy (i.e., $I_i^-=0$) for all months of the horizon. Before discussing that modification, an explanation of the meaning of the two "percentage penalty costs" reported in Table 2 will be given.

Percentage Penalty Cost

In comparing costs (or profits), a relative measure of variation from some standard or benchmark is often more meaningful than the comparison of absolute total costs. This study employs the relative measure of solution accuracy used in similar past studies [17, 20]. This relative measure has been called the "percentage penalty" cost.

The "percentage penalty" cost is the percentage difference between two costs. The first of these costs is the total cost obtained from solving an n-month model (e.g., n=6) for T-consecutive periods (e.g., T=24 months), and the second cost is the total cost resulting from the application of a long-horizon length model (e.g., n=11 or 12 months) for the same T-consecutive periods(i.e., T=24 months). It is assumed that the long-horizon model has a cost of 100 percent

Table 2.

Comparision of 24-Month Simulations with and without shortages

Horizon Length (Months)	Original Model with Shortages (24-Month Percentage Penalty Cost)[a]	I_T (Units)	P_T/t_T (Units/Day)	Model Without Shortages (24-Month Percentage Penalty Cost)[a]	I_T (Units)	P_T/t_T (Units/Day)
1	Infeasible	N/A	N/A	Infeasible	N/A	N/A
2	13.87 (14.91)	3,609	95.85	Infeasible	N/A	N/A
3	8.70 (9.30)	4,578	108.43	1.32 (1.84)	5,294	129.15
5	.59 (1.11)	5,300	129.01	.86 (.90)	5,738	118.01
6	.13 (.18)	5,882	136.33	.72 (.66)	5,963	129.99
7	.07 (.12)	5,738	118.01	.50 (.45)	5,963	129.99
11	.00[b]	5,830	122.87	.49 (.44)	5,963	129.99

[a]The first percentage penalty cost given is calculated from average cost and the second percentage given in parentheses is calculated from adjusted total cost.

[b]Total cost of Model 1 with an 11-month horizon equals zero percentage penalty cost, therefore the I_T and P_T/t_T of that model were the optimal T-period ending conditions.

and consequently a percentage penalty cost of zero percent.

The optimal horizon lengths used in past studies were either 10 or 12 months long [17, 20]. The optimal horizon length used in this study is 11 months.

In calculating percentage penalty costs, the following conditions hold:

1. the models are applied over the same T periods,
2. the models all face the same demand for production over those T periods, and
3. the decisions of the T periods are evaluated with the same cost parameters except that in one modification the shortage strategy

is eliminated.

Ending Condition Adjustment

In the simulations of this research, the ending conditions (i.e., I_T and P_T/t_T) of the last month of the simulated periods are not equal. Solutions to different models often resulted in different T-th period ending inventory levels and production rates (i.e., different s_T, s). Consequently, it would be misleading to directly compare their T-period solution costs. In making comparisons it is necessary to adjust solutions so that they reflect the same ending conditions.

Two percentage penalty costs are presented in Table 2. The first percentage penalty cost is a percentage based on the average cost of producing one unit of production over the T-periods simulated, regardless of differences in ending conditions. The second percentage penalty cost, which will be shown in parentheses below the first, reflects the average cost after adjusting ending conditions to equal the inventory level and production rate of the optimal horizon length model (see note b of Table 2). As shown in Table 2, for a given horizon length and model, the two different percentage penalty costs are very nearly equal; it is therefore evident that differences in the ending conditions of the T periods are not distorting the relative cost of using each model-horizon length combination.

A Comparison With and Without Shortages

The original model without the allowance of shortages was simulated in this paper for several reasons. First, it appeared from the behavior of the original model that significant reductions in the percentage penalty costs of short-horizon solutions would be possible through the elimination of the shortage strategy (i.e., $I_n \geq \check{I}_n$). Secondly, it was felt that the modified model's percentage penalty costs would be much less sensitive to the length of the planning horizon. The reductions in both the percentage penalty cost and horizon sensitivity result from forcing the ending inventory of a horizon (I_n) to be greater than or equal to $\check{I}_n$. Previously, with the allowance of shortages it was noted that the inventory levels of the last month (i.e., I_n) were reduced to the lowest level possible which was the point of maximum shortages (i.e., I_n'). By forcing I_n to be greater than or equal to $\check{I}_n$, the ending inventory would better approximate the horizon's long-run optimal ending conditions. While an ending inventory equal to $\check{I}_n$ may not be the long-run optimal, it was on the average, for short-horizon solutions, substantially closer to the optimum than I_n'.

Unfortunately, by eliminating the shortage strategy completely, the inventory level of every month of the horizon is forced to be greater than or equal to $\check{I}_i$. This constraining of shortages throughout the planning horizon, instead of just the last month, is dysfunctional if shortages are a necessary or lower cost strategy during the first n-1 months of the horizon.

As shown in Table 2, the three-month percentage penalty cost of the model without shortages is significantly less than that of the original model (i.e., 1.32% vs 8.7%), even though shortages are eliminated as a strategy in all months. Unfortunately, with short horizons, infeasibilities were obtained with the original model using a one-month planning horizon, while the modification with its tighter inventory constraints (i.e., $I_n \geq \check{I}_n$) has infeasible solutions with both one- and two-month planning horizons. These infeasibilities occur because both models fail to anticipate seasonal demands which are in excess of amounts that can be supplied from regular and overtime production and inventory. These infeasibilities result in the termination of the simulations and therefore the 24-month total costs are not available for making comparisons.

The 5-, 6-, 7-, and 11-month percentage penalty costs of the original model are lower than those of the modified model. The increase in the modified model's percentage penalty cost results from its inability to use the shortage strategy as a relatively less expensive means of meeting peak seasonal demands. While its longer horizon percentage penalty costs are higher, the difference is not that significant;

consequently, the model without shortages is much less sensitive to the length of the planning horizon than the original model.

A Comparison of Decisions

The criteria used to evaluate the performance of the different solutions of the two models of this paper are explicit. The percentage penalty cost provides a relative measure of the accuracy of the first-month decisions. While the decisions of the first month are often the most critical with respect to current-month operating costs, most meaningful planning is dependent upon the optimality of all the decisions of the horizon. While the use of percentage penalty cost does not provide a measure of the optimality of all of all of the decisions of a single application of an n-month model, it does provide an indirect measure.

The modified model's three-month percentage penalty costs are significantly lower than the original model because the trail-offs in production and inventory levels of the later months witnessed in the original model does not occur. A single application of the modified model results in more accurate total-horizon solutions than those of the original model. Despite the low difference in percentage penalty costs for long horizons, the decisions of the total horizon using the modified model are much more accurate than those of the unmodified model.

Other Model Modifications

The model modification simulated here is only one of several which have been simulated using this model. Many other possible modifications to the ending conditions (i.e., parameters and costs) could be simulated; for a discussion of others refer to [17].

The elimination of the shortage strategy for all months of the planning horizon is a simple modification; yet it had significant beneficial effects upon both the percentage penalty cost of short-horizon solutions and the decisions of the later months of long- and short-horizon solutions. A simple extension of that modification has been simulated. It involved not the elimination of the shortage strategy but the avoidance of shortages in just the last month of the horizon. By establishing an excessive penalty cost in the last period of the horizon, both the infeasibilities of the two shortest-horizon solutions (i.e., one- and two-month models) and the less optimal solutions of the longer-horizon solutions (i.e., 5-, 6-, 7-, and 11-months) were eliminated.

VII. SUMMARY AND CONCLUSIONS

This paper has explored the relationship between the horizon length and the costs of applying a general linear programming aggregate operations model. While simulating the use of the model with and without the allowance of shortage, it was noted that substantial reductions in the total costs of solutions were possible through the elimination of the shortage strategy. The results of the study demonstrated the possible reductions in total costs achieved through the use of an approximation of a horizon's long-run optimal ending condition. Solutions were therefore much less sensitive to the length of the horizon included in the model; consequently the decisions of the modified model were much more accurate than those of the original.

In addition to the above results, this paper has briefly introduced the use of MPSX in solving and simulating multiperiod decision models. The procedures of MPSX aided significantly in the formulation, solution, and simulation of this model and several modifications to it.

A natural extension of the procedures and methodology of this research is one involving the application of linear programming under uncertainty. Often the users of linear programming face considerable uncertainty with respect to the values of the LP parameters. Under such situations post-optimality analysis or simulation procedures can be used to investigate the effects of parametric variances upon solution optimality.

In solving multistage decision models, the user is often concerned with the effect of uncertainty upon the optimality of the current (i.e., first) period decisions. Unfortunately, as the complexity of a model increases, and consequently the number of uncertainties, the applicability of LP post-optimality procedures becomes limited. Because of such limitations, Monte Carlo simulation methods may offer a viable solution technique.

The MPSX procedure described here could also be used under conditions of uncertainty. The algorithm developed previously remains as shown, wherein only the content and purpose of the external FORTRAN program need be modified. Through the use of REVISE and the external FORTRAN subprogram, the effects of variances in parameters may be studied more meaningfully.

REFERENCES

1. Buffa, E.S. and Taubert, William H., *Production-Inventory Systems: Planning and Control*. Homewood, Illinois: Richard D. Irwin, Inc., 1972.
2. Cohen, Claude, Reagan, Michael D., and Stein, Jack R., "Design of an Optimization System for University Use," presented at the 8th International Mathematical Programming Symposium, August 27-31, 1973, Stanford University, California.
3. Dantzig, G.B., *Linear Programming and Extensions*. Princeton, N.J.: Princeton University Press, 1963.
4. Eilon, Samuel, "Five Approaches to Aggregate Production Planning," *AIIE Transactions*, 7:118-176 (June 1975).
5. Gorenstein, Samuel, "Planning Tire Production," *Management Science*, 17:B72-82 (October 1970).
6. Hanssmann, Fred and Hess, Sidney W., "A Linear Programming Approach to Production and Employment Scheduling," *Management Technology*, 1:46-51 (January 1960).
7. Holt, Charles, Modigliani, Franco, Muth, J.E. and Simon, H.A., *Planning Production, Inventories, and Work Force*. Englewood

Cliffs, N.J.: Prentice-Hall, Inc., 1960.

8. Holt, Charles, Modigliani, Franco and Simon, Herbert A., "A Linear Decision Rule for Production and Employment Scheduling," *Management Science*, 2:1-30 (October 1955).
9. I.B.M. Corporation, *Mathematical Programming System Extended (MPSX), and Generalized Upper Bounding (GUB) Program Description* (SH20-0968-1). White Plains, N.Y.: I.B.M. Corporation, 1971.
10. I.B.M. Corporation, *Control Language User's Manual* (SH20-0932). White Plains, N.Y.: I.B.M. Corporation, 1971.
11. Kwak, N.K., *Mathematical Programming with Business Applications*. New York: McGraw-Hill Book Co., 1973.
12. Kwak, N.K. and DeLurgio, S.A., "Optimal Production Scheduling and Inventory Control Under Non-Stationary Demands," in *Studies in Linear Programming*, edited by Harvey M. Salkin and Jahar Saha, Amsterdam: North-Holland Publishing Company, 1975.
13. Magee, John F. *Production Planning and Inventory Control*. New York: McGraw-Hill Book Co., 1958.
14. McGarrah, R.E., *Production and Logistics Management: Text and Cases*. New York: John Wiley and Sons, Inc., 1963.
15. Modigliani, Franco and Hohn, F.E., "Production Planning Over Time and the Nature of the Expectations and Planning Horizon," *Econometrica*, 23:46-66 (January 1955).
16. Ricketts, Donald E. and Zimmer, Robert K., "A Dynamic Optimization Model for Planning in a Multi-Product Environment," *Decision Sciences*, 6:274-283 (April 1975).
17. Sikes, Thomas, *The Search Decision Rule Applied to Aggregate Planning: Improved Efficiency and Stochastic Extension*. Unpublished Ph.D. Dissertation, University of California at Los Angeles (August 1970).
18. Silver, Edward A, "A Tutorial on Production Smoothing and Work Force Balancing," *Operations Research*, 15:985-1010 (November-December 1967).

19. Taubert, William H., "A Search Decision Rule for the Aggregate Scheduling Problem," *Management Science*, 14: B343-359(February 1968).

20. Taubert, William H., *The Search Decision Rule Approach to Operations Planning.* Unpublished Ph.D. Dissertation, University of California at Los Angeles, 1968.

21. Van de Panne, C. and Bosje, P., "Sensitivity Analysis of Cost Coefficient Estimates: The Case of Linear Decision Rules for Employment and Production," *Management Science*, 9:82-105 (October 1962).

22. Vergin, Roger C., "Production Scheduling Under Seasonal Demand," *Journal of Industrial Engineering*, 17:260-266 (May 1966)

23. Winters, Peter R., "Constrained Inventory Rules for Production Smoothing," *Management Science*, 8:470-581 (July 1962).